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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ... LAGRANGE

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A Variational Principle of Hydromechanics

S. DROBOT & A. RYBARSKI

Communicated by C. TRUESDELL

I. Introduction

1. The purpose of this paper is to reveal the part played in variational principles of hydromechanics by a certain group of infinitesimal transformations of the fields of density and velocity. This group, called here the hydromechanical variation and originating from some elementary requirement concerning variation of matter, seems to be essential in variational principles of hydromechanics, not only in deducing from them the equations of motion, jump conditions included, but also in studying the background of the conservation laws, the vorticity theorems included.

So as to obtain such conservation laws, here we shall deal only with extremal variational principles, which assert that the equations of motion render the action an extremum. (They do not include flows in which dissipative forces act, *e.g.*, flows of a viscous fluid.) The action will be taken as an integral, over a given time-space region, of any given Lagrangean depending on the fields of density and velocity only, these being considered as functions of the time-space point. The Lagrangean will not be *a priori* restricted to the usual, form, *viz.*, the difference between the kinetic and the potential energy. However, it is not our aim to find more hydromechanical cases which can be included by these principles or to study their range of relevance. We desire rather to ascertain which mathematical facts cause the equations of motion to follow from the variational principle, independently of the particular form of Lagrangean assumed; which facts lead to the conservation laws; and — as will be shown to be connected — which ones particularize the form of the Lagrangean.

Namely, by suitably embedding the extremal principle of hydromechanics into the general framework of the calculus of variations, it will be proved that the invariance of the action with respect to a Galilean subgroup of the hydromechanical variation leads to the conservation laws of matter, of impulse and of angular momentum and restricts the form of the Lagrangean; however, the energy equation and — as seems to be more interesting and new — the vorticity conservation laws depend neither on this Galilean subgroup nor on the particular form of the Lagrangean but are subject to another intrinsic subgroup of the hydromechanical variation.

Of course, not all results of this paper are claimed to be new. In appropriate places, some of the numerous existing related works will be quoted, but the list of them is by no means complete. We are greatly indebted to Professor C. TRUESDELL for valuable discussion of the first draft of this paper.

2. In order to introduce the basic idea of the hydromechanical variation, we start by recalling that when in variational principles of hydromechanics we use Eulerian or spatial variables, some peculiar features arise which are not so evident when using Lagrangean or material variables. As in many earlier works, we find it instructive to compare the variational principles for continuous media with the principles valid in the mechanics of discrete material points.

As typical of extremal principles for a system of N discrete material points with masses m_a, $a=1, 2, \ldots, N$, coordinates x_a^i, $i=1, 2, 3$, and potential energy U, let us consider HAMILTON's principle:

$$\delta \int_{t_0}^{t_1} dt \left\{ \sum_{a=1}^{N} \frac{1}{2} m_a \left[\left(\frac{d x_a^1}{dt}\right)^2 + \left(\frac{d x_a^2}{dt}\right)^2 + \left(\frac{d x_a^3}{dt}\right)^2 \right] - U \right\} = 0 \tag{1.1}$$

for all variations $\delta x_a^i(t)$ vanishing at the ends of the time interval $t_0 \leqq t \leqq t_1$. We could, indeed, consider a more general integrand depending arbitrarily on all the variables occurring, but this would not illumine the questions in which we are interested.

In hydromechanics formulated exclusively in terms of the Lagrangean or material variables a, b, c, the principle is merely an evident analogy to (1.1)[1]. It suffices to take instead of the finite sum over the indices a the integral over a given region V_0 of the triplets a, b, c; instead of the masses m_a the element $\varrho_0(a, b, c)\, dV_0$, in which ϱ_0 denotes the density of the fluid at the time t_0; instead of the coordinates x_a^i the functions $x^i(t, a, b, c)$, $i=1, 2, 3$; and to require, with a suitably chosen potential energy U, that

$$\delta \int_{t_0}^{t_1} dt \iiint_{V_0} dV_0 \left\{ \frac{1}{2} \varrho_0 \left[\left(\frac{d x^1}{dt}\right)^2 + \left(\frac{d x^2}{dt}\right)^2 + \left(\frac{d x^3}{dt}\right)^2 \right] - U \right\} = 0 \tag{1.2}$$

for all variations $\delta x^i(t, a, b, c)$ vanishing at the ends t_0, t_1 of the time interval. Guided in the same manner by the more general form of (1.1) with any given Lagrangean, we could obtain a corresponding generalization of (1.2)[2].

It is a different matter when we use exclusively the Eulerian or spatial variables and regard the velocity $\vec{v}(t, x^1, x^2, x^3)$ and the density $\varrho(t, x^1, x^2, x^3)$ as fields defined at the point (x^1, x^2, x^3) in a given spatial region V_t and at the time t in a given interval $t_0 \leqq t \leqq t_1$. Should we try, purely formally, to write down the continuous analogue to (1.1) by taking, *e.g.*, the potential energy ε as a given function, and by requiring

$$\delta \int_{t_0}^{t_1} dt \iiint_{V_t} dV_t \left(\tfrac{1}{2} \varrho\, v^2 - \varepsilon\right) = 0 \tag{1.3}$$

for all variations $\delta\varrho$ and $\delta\vec{v}$ vanishing for t_0, t_1 and on the boundary S_t of the region V_t, we should get from such a "principle" $\vec{v}=0$ as equations of "motion". This cannot be avoided by taking a more general form of the Lagrangean, provided that it depends on t, x^1, x^2, x^3 and on the fields $\varrho, \vec{v}$ only. The reason, of course, is that not all fields $\varrho, \vec{v}$ should be allowed in competition. It would seem to be

[1] *Cf. e.g.* HELLINGER [1914], p. 656, formula (25).

[2] *Cf.* HELLINGER [1914], p. 656, formula (26).

enough to take the continuity equation

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v}) = 0 \tag{1.4}$$

expressing the conservation of matter, as a side condition to (1.3), or, what is mathematically the same, by introducing some Lagrangean multiplier $\varphi(t, x^1, x^2, x^3)$ as an additional function to be determined from the principle, to require that

$$\delta \int_{t_0}^{t_1} dt \iiint_{V_t} dV_t \left\{\left(\frac{1}{2}\varrho v^2 - \varepsilon\right) + \varphi\left(\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v})\right)\right\} = 0 \tag{1.5}$$

for all variations $\delta\varrho$, $\delta\vec{v}$ and $\delta\varphi$ vanishing on S_t and for t_0, t_1. However, the equations of motion which follow from such a principle, *viz.*

$$\frac{1}{2}v^2 + \frac{\partial \varepsilon}{\partial \varrho} + \frac{\partial \varphi}{\partial t} = 0, \qquad \vec{v} = \operatorname{grad}\varphi, \qquad \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v}) = 0$$

pertain only to irrotational flow, φ being the potential of the velocity[3]. This result would not be improved by taking instead of the primary term $\frac{1}{2}\varrho v^2 - \varepsilon$ any other function depending on t, x^1, x^2, x^3 and ϱ, $\vec{v}$ only.

To obtain more general equations of motion one introduces[4] three additional functions $\varphi(t, x^1, x^2, x^3)$, $\lambda(t, x^1, x^2, x^3)$, $\mu(t, x^1, x^2, x^3)$ and requires that

$$\delta \int_{t_0}^{t_1} dt \iiint_{V_t} dV_t \left\{\left(\frac{1}{2}\varrho v^2 - \varepsilon\right) + \varphi\left(\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v})\right) + \lambda\left(\frac{\partial \mu}{\partial t} + \vec{v}\operatorname{grad}\mu\right)\right\} = 0 \tag{1.6}$$

for all variations $\delta\varrho$, $\delta\vec{v}$ and $\delta\varphi$, $\delta\lambda$, $\delta\mu$ vanishing on S_t and for t_0, t_1. The six equations which follow from this principle are

$$\frac{1}{2}v^2 + \frac{\partial \varepsilon}{\partial \varrho} + \frac{\partial \varphi}{\partial t} - \lambda\frac{\partial \mu}{\partial t} = 0, \qquad \vec{v} = \operatorname{grad}\varphi - \lambda\operatorname{grad}\mu,$$

$$\frac{\partial \mu}{\partial t} + \vec{v}\operatorname{grad}\mu = 0, \qquad \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v}) = 0.$$

The first of them is BERNOULLI'S equation; the next two are CLEBSCH'S equations, and $\lambda = \text{const}$, $\mu = \text{const}$ are equations of the vortex lines[5]. By eliminating the three functions φ, λ, μ, we get the usual equations of continuity and momentum for the primary fields ϱ, $\vec{v}$. Plainly the principle (1.6) can be generalized by taking instead of the term $\frac{1}{2}\varrho v^2 - \varepsilon$ any other function depending on t, x^1, x^2, x^3 and ϱ, $\vec{v}$; however, the additional terms will remain the same.

It seems worthwhile to discover the part played by these additional terms, the introduction of which has been justified heretofore rather by the final success in getting formally the correct equations of motion than by any interpretation.

3. To this end, let us analyse another approach, in which, however, both material and spatial variables are used. Namely, the continuity equation is used

[3] *Cf.* BATEMAN [1930], pp. 816–825.
[4] CLEBSCH [1859], pp. 1–10. *Cf.* also BATEMAN [1944], pp. 164–166.
[5] *Cf.*, *e.g.*, LAMB [1906], pp. 233–234.

in a material form[6],

$$\varrho(t, x^1, x^2, x^3) = \varrho_0 \frac{\partial(t, a, b, c)}{\partial(t, x^1, x^2, x^3)}, \tag{1.7}$$

so that

$$\iiint_{V_t} \varrho \, dx^1 dx^2 dx^3 = \iiint_{V_0} \varrho_0 \, da \, db \, dc$$

where V_t is the region of places (x^1, x^2, x^3) occupied at time t by the particles a, b, c constituting the region V_0; taking a suitable expression $\mathfrak{M}$ for the potential energy, one states, *e.g.*, Hamilton's principle by requiring that

$$\delta \int_{t_0}^{t_1} dt \iiint_{V_t} dV_t \left\{\varrho\left[\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2\right] - \mathfrak{M}\right\} = 0 \tag{1.8}$$

for all variations δx^i, $i = 1, 2, 3$ which vanish at the ends of the time interval, provided, however, that the total variation of the mass satisfies the condition

$$\Delta \iiint_{V_t} \varrho \, dV_t = 0, \tag{1.9}$$

which leads[7] to the following condition for variation of the density:

$$\Delta \varrho = -\varrho \frac{\partial(\delta x^i)}{\partial x^i}, \tag{1.10}$$

where the summation convention is used. While spatial variables are used in (1.10), in evaluating the variations of the velocities in (1.8) one takes $\delta\left(\frac{dx^i}{dt}\right) = \frac{d\,\delta x^i}{dt}$, which implies use of material variables. The principle can be generalized by taking as integrand an arbitrary function of the same variables[8], the conditions (1.9) and (1.10) being retained.

Just these conditions (1.9) or (1.10) will be proved to be essential in deducing the equations of motion from the variational principle. In fact, a special use of them is implicitly included in the introduction of the functions λ, φ, μ into the principle (1.6).

In this paper we shall deal with the fields $\varrho, \vec{v}$ exclusively, using only field variables t, x^1, x^2, x^3 and not introducing additional functions, and we shall impose some conditions on the variations of the fields which constitute a generalization of (1.10). The physical background of these conditions can be found out by analysing more carefully some assumptions adopted, usually tacitly, in principles appropriate to discrete material points. Namely, one always assumes that

$$\delta t = 0 \quad \text{and} \quad \delta m = 0, \tag{1.11}$$

[6] *Cf.* Hellinger [1914], p. 609, formula (7); also Lichtenstein [1929], pp. 361 to 365, particularly formulae (7), (8), (10) and (15).

[7] *Cf.* Lichtenstein [1929], p. 343, formula (78); also Hellinger [1914], p. 609, formula (8′), where a different notation is used.

[8] In gas dynamics the internal energy is assumed to depend not only on the density but also on a *third* variable, such as the temperature. The problem of formulating a purely spatial variational principle for such gas flows is one of classical difficulty. C. Truesdell has informed us that this problem has been solved recently by C. C. Lin, correcting insufficient earlier formulations by Eckart and Herivel.

and these are counterparts of (1.10). Further, one assumes that

$$\frac{dm}{dt} = 0, \tag{1.12}$$

to which the continuity equation (1.4) or (1.7) corresponds. These conditions are logically independent and, although elementary, by no means trivial. We shall employ them in generalized form. Instead of the pair of assumptions (1.11) we shall adopt the continuous counterpart to the following:

$$\delta t = 0 \quad \text{implies} \quad \delta m = 0, \tag{1.13}$$

and instead of (1.12), the continuous counterpart to the condition

$$\delta\left(\frac{dm}{dt}\right) = 0, \tag{1.14}$$

which makes possible inclusion of variational principles for points with variable masses and, in hydromechanics, flows with sources. Formulation of counterparts to the conditions (1.13) and (1.14) in terms of field variables leads to the definition of the hydromechanical variation in § II, 1 below.

4. We now explain the fields we shall use. In the Euclidean-Galilean four-dimensional space $\mathfrak{X}_4$ of the points x with the coordinates x^α, $\alpha = 0, 1, 2, 3$, by taking x^0 as the time t and x^i, $i = 1, 2, 3$, as the space-like coordinates, we consider the four-dimensional vector fields $p(x)$ with the components $p^\alpha(x)$, in which p^0 is the density ϱ, and p^i, $i = 1, 2, 3$, are the impulses ϱv^i, where v^i denotes the component of the velocity $\vec{v}$.

The notion of the mass contained at the time t in the volume V_t will be extended as follows. Let $\sigma^\bullet$ be any 3-dimensional hypersurface contained in $\mathfrak{X}_4$, and let $d\tau_\alpha$ denote the oriented element on $\sigma^\bullet$, *i.e.*,

$$d\tau_\alpha = e_{\alpha\beta\gamma\delta}\, \underset{1}{dx^\beta}\, \underset{2}{dx^\gamma}\, \underset{3}{dx^\delta} \tag{1.15}$$

where $e_{\alpha\beta\gamma\delta} = e^{\alpha\beta\gamma\delta}$ is Ricci's symbol, having the value 0 if two of the indices are equal, $+1$ if the permutation of the indices is even, and -1 if the permutation is odd, and where $\underset{1}{dx^\beta}, \underset{2}{dx^\gamma}, \underset{3}{dx^\delta}$, are three linearly independent vectors lying on $\sigma^\bullet$; thus the vector $d\tau_\alpha$ is normal to the hypersurface $\sigma^\bullet$. The mass contained on $\sigma^\bullet$ will be represented by the integral $\int_{\sigma^\bullet} d\tau_\alpha p^\alpha$, called also the complete matter-flow. In the particular case when the hypersurface $\sigma^\bullet$ is the space-like three-dimensional volume V, we have $d\tau_0 = dV$, $d\tau_i = 0$, $i = 1, 2, 3$, and

$$\int_{\sigma^\bullet} d\tau_\alpha p^\alpha = \iiint_V \varrho\, dV \tag{1.16}$$

reduces to the usual mass. More generally, when $\sigma^\bullet$ is closed and consists of V_{t_0}, V_t and of the moving two-dimensional boundary S_t of V_t for $t_0 < t < t_1$, then

$$\oint_{\sigma^\bullet} d\tau_\alpha p^\alpha = \iiint_{V_{t_1}} \varrho\, dV_t - \iiint_{V_{t_0}} \varrho\, dV_t + \int_{t_0}^{t_1} dt \iint_{S_t} \varrho \vec{v} \cdot d\vec{S}_t, \tag{1.17}$$

this identity being known[9] as the matter-balance for the moving region. In the general case, when the hypersurface $\sigma^\bullet$ may be open, the complete matter-flow

[9] *Cf.* Cartan [1922], pp. 36–37.

represents a generalization of the notion of the mass contained in $\sigma^\bullet$. By using Gauss' formula

$$\oint_{\tau^\bullet} d\tau_\alpha p^\alpha = \int_\tau d\tau\, \partial_\alpha p^\alpha, \tag{1.18}$$

in which τ denotes any four-dimensional region contained in $\mathfrak{X}_4$, and $\tau^\bullet$ its boundary, $\partial_\alpha = \partial/\partial x^\alpha$, we see that

$$\partial_\alpha p^\alpha = \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \vec{v}) \tag{1.19}$$

represents the density of the sources of matter existing in τ.

The action W will be assumed to have the form

$$W = \int_\tau d\tau\, L(x, p(x)), \tag{1.20}$$

in which the Lagrangean L is any given function depending on x and $p(x)$ only. This form includes, *e.g.*, the usual one for barotropic flow of an inviscid fluid, *viz.*, as the special case when

$$L = \frac{1}{2p^0}\{(p^1)^2 + (p^2)^2 + (p^3)^3\} - \varepsilon(p^0) - p^0 U(x), \tag{1.21}$$

where ε is the potential energy, which depends only on $p^0 = \varrho$, and U is the potential of the external forces.

II. The variational principle

1. In this section we formulate a variational principle of hydromechanics and deduce from it the equations of motion, whatever the assumed form of the Lagrangean may be. In this way we hope to show which facts are responsible for the derivation of these equations only.

First we prepare some notions and formulas concerning variations. Let $\mathfrak{P}$ be a functional space of vector-valued functions $p(x)$ with the components $p^\alpha(x)$ supposed to be sufficiently regular in $\mathfrak{X}_4$. Consider the following infinitesimal transformations of the spaces $\mathfrak{X}_4$ and $\mathfrak{P}$ into themselves:

$$\bar{x} = x + \delta x(x), \tag{2.1}$$

and

$$\bar{p} = p + \delta p(x). \tag{2.2}$$

More exactly,

$$\delta x(x) = e\xi(x) + o(e), \qquad \delta p(x) = e\pi(x) + o(e), \tag{2.3}$$

where ξ and π are arbitrary functions belonging to the space $\mathfrak{P}$, e is a scalar parameter, and $o(e)/e \to 0$ as $e \to 0$. The functions $\delta p(x)$ are called the local variations of the field p, and the functions Δp given by

$$\Delta p = \delta p + \partial_\beta p \cdot \delta x^\beta, \tag{2.4}$$

where $\partial_\beta = \partial/\partial x^\beta$, are called the total variations of p. To within infinitesimals of first order we have

$$\Delta p = \bar{p}(\bar{x}) - p(x). \tag{2.5}$$

The operation

$$\Delta \int_{\sigma^\bullet} d\tau_\alpha p^\alpha = e\left\{\frac{d}{de}\int_{\bar\sigma^\bullet} d\tau_\alpha \bar p^\alpha(\bar x)\right\}_{e=0}, \tag{2.6}$$

where $\bar\sigma^\bullet$ denotes the hypersurface obtained from $\sigma^\bullet$ by the transformation (2.1), will be called the total variation of the complete matter-flow. We have the identity

$$\Delta \int_{\sigma^\bullet} d\tau_\alpha p^\alpha = \int_{\sigma^\bullet} d\tau_\alpha\{\delta p^\alpha - \partial_\beta(p^\beta\,\delta x^\alpha - p^\alpha\,\delta x^\beta) - \partial_\beta p^\beta\cdot\delta x^\alpha\}, \tag{2.7}$$

which can be proved by calculating the right-hand side of (2.6):

$$\begin{aligned} e\left\{\frac{d}{de}\int_{\bar\sigma^\bullet} d\tau_\alpha \bar p^\alpha\right\}_{e=0} &= e\left\{\frac{d}{de}\int_{\sigma^\bullet} d\tau_\alpha \left(\frac{\partial\bar x}{\partial x}\right)^\alpha_\beta \bar p^\alpha(\bar x)\right\}_{e=0} \\ &= e\left\{\int_{\sigma^\bullet} d\tau_\alpha\left[\frac{d}{de}\left(\frac{\partial\bar x}{\partial x}\right)^\alpha_\beta \bar p^\beta + \left(\frac{\partial\bar x}{\partial x}\right)^\alpha_\beta \frac{d}{de}(p^\beta + \Delta p^\beta)\right]\right\}_{e=0} \\ &= \int_{\sigma^\bullet} d\tau_\alpha\left\{e\left[\frac{d}{de}\left(\frac{\partial\bar x}{\partial x}\right)^\alpha_\beta\right]_{e=0} p^\beta + \delta^\alpha_\beta\,\Delta p^\beta\right\}. \end{aligned}$$

Here $\left(\frac{\partial\bar x}{\partial x}\right)^\alpha_\beta$ denotes a minor of the Jacobian $\frac{\partial\bar x}{\partial x}$ of the transformation (2.1), and therefore

$$\left(\frac{\partial\bar x}{\partial x}\right)^\alpha_\gamma \partial_\beta \bar x^\gamma = \delta^\alpha_\beta\,\frac{\partial\bar x}{\partial x},$$

δ^α_β being the Kronecker delta, whence by differentiating with respect to e and then putting $e=0$ we have

$$e\left\{\frac{d}{de}\left(\frac{\partial\bar x}{\partial x}\right)^\alpha_\gamma\right\}_{e=0}\delta^\gamma_\beta + \delta^\alpha_\gamma\,\partial_\beta\,\delta x^\gamma = \delta^\alpha_\beta\,\partial_\gamma\,\delta x^\gamma.$$

Thus, finally, by (2.4) we get (2.6).

The functional

$$\delta W = e\left\{\frac{d}{de}\int_\tau d\tau\, L(x,\bar p(x))\right\}_{e=0} \tag{2.8}$$

is called the local variation of the action W given by (1.20), and

$$\Delta W = e\left\{\frac{d}{de}\int_{\bar\tau} d\tau\, L(\bar x,\bar p(\bar x))\right\}_{e=0} \tag{2.9}$$

where $\bar\tau$ denotes the region obtained from τ by (2.1), is called the total variation of the action. We have[10]

$$\Delta W = \int_\tau d\tau\left\{\frac{\partial L}{\partial p^\alpha}\,\delta p^\alpha + \partial_\alpha(L\,\delta x^\alpha)\right\}. \tag{2.10}$$

So far, the variations of the field $p(x)$, both local and total, have been arbitrary infinitesimally small functions belonging to the whole space $\mathfrak{P}$. Now we impose on them the following conditions:

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[10] *Cf.*, *e.g.*, COURANT-HILBERT [1930], vol. 1, Ch. IV, § 11, p. 8.

1° For every hypersurface $\sigma^\bullet$ for which $d\tau_\alpha\,\delta x^\alpha = 0$,

$$\Delta \int_{\sigma^\bullet} d\tau_\alpha\, p^\alpha = 0. \tag{2.11}$$

2° The variation δp^α shall satisfy the equation

$$\partial_\alpha\, \delta p^\alpha = 0. \tag{2.12}$$

The variations $\delta_0 p^\alpha$ satisfying these conditions will be called the local *hydromechanical variations* of the field p. According to (2.4) we write

$$\Delta_0 p^\alpha = \delta_0 p^\alpha + \partial_\beta p^\alpha \cdot \delta x^\beta \tag{2.13}$$

and call $\Delta_0 p^\alpha$ the total hydromechanical variation of the field p.

Theorem 1. *All local hydromechanical variations are of the form*

$$\delta_0 p^\alpha = \partial_\beta (p^\beta\, \delta x^\alpha - p^\alpha\, \delta x^\beta), \tag{2.14}$$

where the δx^α are arbitrary infinitesimal functions belonging to the space $\mathfrak{P}$.

Proof. In order to prove that the condition (2.13) is necessary, let us substitute formula (2.7) in equation (2.11):

$$\Delta \int_{\sigma^\bullet} d\tau_\alpha\, p^\alpha = \int_{\sigma^\bullet} d\tau_\alpha \{\delta_0 p^\alpha - \partial_\beta (p^\beta\, \delta x^\alpha - p^\alpha\, \delta x^\beta) + \partial_\beta p^\beta \cdot \delta x^\alpha\} = 0.$$

From the condition 1° it follows that $d\tau_\alpha\, \delta x^\alpha = 0$ implies

$$d\tau_\alpha \{\delta_0 p^\alpha - \partial_\beta (p^\beta\, \delta x^\alpha - p^\alpha\, \delta x^\beta)\} = 0;$$

thus we have

$$\delta_0 p^\alpha - \partial_\beta (p^\beta\, \delta x^\alpha - p^\alpha\, \delta x^\beta) = K\, \delta x^\alpha,$$

K being some factor of proportionality. But the condition 2° requires that $\partial_\alpha (K\, \delta x^\alpha) = 0$ for all δx^α, whence $K = 0$, and (2.14) follows. The condition (2.14) is also sufficient. In fact, from (2.14) it follows that (2.12) holds identically, and from (2.7) that the condition 1° is also satisfied, which completes the proof.

The formula (2.14) defines an infinitesimal group of transformations of the vector field $p(x)$, depending on arbitrary functions $\delta x^\alpha(x)$.

We make a further comment on the conditions defining the hydromechanical variations. The condition 1° is an appropriate counterpart to the condition (1.13). In fact, in the particular case when $d\tau_0 = dV$, $d\tau_i = 0$, $i = 1, 2, 3$, the condition 1° requires that when $\varrho\, dV\, \delta t = 0$, and hence $\delta t = 0$, (1.9) must hold. From (2.14) it follows also that then (1.10) holds. But this is only a special case of the general formulae (2.11) and (2.14), *viz.*, that obtained by putting $\alpha = 0$, $\delta x^0 = \delta t = 0$. In the general case, the condition 1° requires that for variations δx^α tangential to the hypersurface $\sigma^\bullet$ the total variation of the complete matter-flow contained on $\sigma^\bullet$ shall vanish, corresponding to (1.13).

The condition 2°, which is equivalent to $\delta(\partial_\alpha p^\alpha) = 0$, requires no local variation of the source density; this corresponds to (1.14), being a generalization of it appropriate to continuous media. It should be stressed that the condition 2° makes no use of the continuity equation

$$\partial_\alpha p^\alpha = 0, \tag{2.15}$$

which is thus superfluous in deducing equations of motion from the variational principle.

It may be noted also that the concept of hydromechanical variation can be introduced also if $\mathfrak{X}_4$ is a Riemannian space. It suffices to write the symbol of the covariant derivative instead of ∂_β, as can be verified by calculation[11].

2. After these preparations we formulate the variational principle:

For all δx^α vanishing on the boundary of the region τ,

$$\Delta W = \Delta \int_\tau d\tau\, L(x, p(x)) = 0, \tag{2.16}$$

provided that the $\delta_0 p^\alpha$ are hydromechanical variations.

We now deduce the equations of motion from this principle. By the formulae (2.10) and (2.14) we have identically in all variables

$$\begin{aligned}\Delta W &= \int_\tau d\tau \left\{\frac{\partial L}{\partial p^\alpha}\, \partial_\beta (p^\beta\, \delta x^\alpha - p^\alpha\, \delta x^\beta) + \partial_\alpha (L\, \delta x^\alpha)\right\} \\ &= \int_\tau d\tau\, \partial_\beta (T^\beta_\alpha\, \delta x^\alpha) - \int_\tau d\tau\, \psi_\alpha\, \delta x^\alpha,\end{aligned} \tag{2.17}$$

where

$$T^\beta_\alpha = p^\beta \frac{\partial L}{\partial p^\alpha} + \delta^\beta_\alpha \left(L - p^\gamma \frac{\partial L}{\partial p^\gamma}\right), \tag{2.18}$$

and

$$\psi_\alpha = p^\beta \left\{\partial_\beta \left(\frac{\partial L}{\partial p^\alpha}\right) - \partial_\alpha \left(\frac{\partial L}{\partial p^\beta}\right)\right\}. \tag{2.19}$$

Since the δx^α vanish on the boundary of the region τ, the first integral on the right-hand side in (2.17) vanishes, and since the δx^α are arbitrary in the interior of τ, we obtain from (2.16) the following scheme of equations of motion:

$$\psi_\alpha = 0, \tag{2.20}$$

which can be transformed into

$$p^\beta\, \partial_\beta \left(\frac{\partial L}{\partial p^\alpha}\right) + \partial_\alpha \left(L - p^\gamma \frac{\partial L}{\partial p^\gamma}\right) = \frac{\partial L}{\partial x^\alpha_{ex}}, \tag{2.21}$$

where the sign $\partial L/\partial x^\alpha_{ex}$ denotes the explicit partial derivative, *i.e.*, $\partial L(y, p)/\partial y^\alpha$ for $y^\alpha = x^\alpha$.

The four equations of motion (2.20) are linearly dependent, since identically

$$p^\alpha\, \psi_\alpha = 0. \tag{2.22}$$

(This fact will be discussed in Section III, 3.) Thus the four functions p^α describing the flow satisfy only three independent equations.

The same equations can be also obtained by using the local variation δ instead of the total Δ, *i.e.*, by requiring that $\delta W = 0$.

In the case when the Lagrangean takes the usual form (1.21), we get from (2.20) for $\alpha = 0$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{v}^2\right) + \vec{v}\,\mathrm{grad}\left(\frac{1}{2}\vec{v}^2\right) + \vec{v}\left(\frac{1}{2}\,\mathrm{grad}\, P + \mathrm{grad}\, U\right) = 0$$

[11] A formula analogous to (2.14) is used by Fock [1955] pp. 214–218, particularly formulae (47.35) and (47.36), but deduced in a different way.

and for $\alpha = i = 1, 2, 3$

$$\frac{\partial \vec{v}}{\partial t} + \operatorname{grad}\left(\frac{1}{2}\vec{v}^2\right) - \vec{v} \times \operatorname{rot} \vec{v} + \frac{1}{\varrho} \operatorname{grad} P + \operatorname{grad} U = 0$$

where

$$P = \varrho \frac{d\zeta}{d\varrho} - \varepsilon$$

is the pressure. When $\alpha = 0$, the equations (2.21) give the same Bernoulli equation as above, and for $\alpha = 1, 2, 3$ the usual Euler equations:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \operatorname{grad}) \vec{v} + \frac{1}{\varrho} \operatorname{grad} P + \operatorname{grad} U = 0.$$

All these equations of motion have been deduced under the assumption that the functions p^α are continuous in the whole region τ. However, if there exist some hypersurfaces on which the p^α cease to be continuous, jump conditions for them can also be deduced from the variational principle. For simplicity, but without any essential loss of generality, let us assume that in τ there exists such a single hypersurface $\sigma^\bullet$ on which the functions p^α are not continuous but approach on each side finite and continuously differentiable limits. Let

$$F(x) = 0 \tag{2.23}$$

be the equation of this hypersurface, and let its unit normal vector ν_β, which is proportional to the derivative $\partial_\beta F$, be continuous on $\sigma^\bullet$. We assume that the hypersurface $\sigma^\bullet$ divides the region τ into two parts τ^- and τ^+, and that the normal vector is directed from the part τ^- into the part τ^+. Denoting by $d\tau_\beta^-$ and $d\tau_\beta^+$ the elements of hypersurface directed inside τ^- and τ^+, respectively, and by $d\sigma^\bullet$ their common absolute value, we have

$$d\tau_\beta^- = -\nu_\beta \, d\sigma^\bullet, \qquad d\tau_\beta^+ = \nu_\beta \, d\sigma^\bullet.$$

Consider the expression T_α^β given by (2.18). Let $\overset{-}{T}{}_\alpha^\beta$ denote the limit of T_α^β if we approach the hypersurface $\sigma^\bullet$ from within τ^-, and let $\overset{+}{T}{}_\alpha^\beta$ be the corresponding limit from within τ^+. Then equation (2.16) becomes

$$\Delta \int_{\tau^-} d\tau \, L + \Delta \int_{\tau^+} d\tau \, L = 0,$$

and from (2.17) it follows that inside both regions τ^- and τ^+ the equations of motion (2.20) hold. But on $\sigma^\bullet$

$$\int_{\sigma^\bullet} d\sigma^\bullet \, \nu_\beta \left(\overset{+}{T}{}_\alpha^\beta - \overset{-}{T}{}_\alpha^\beta\right) \delta x^\alpha = 0$$

for all δx^α. Thus, denoting the jump of a function on the hypersurface $\sigma^\bullet$ by square brackets, *e.g.*

$$[T_\alpha^\beta] = \overset{+}{T}{}_\alpha^\beta - \overset{-}{T}{}_\alpha^\beta,$$

we obtain the following scheme of jump conditions:

$$\nu_\beta [T_\alpha^\beta] = 0. \tag{2.24}$$

By substituting from (2.18) we get

$$\left[p^\beta \frac{\partial L}{\partial p^\alpha}\right] \partial_\beta F + \left[L - p^\beta \frac{\partial L}{\partial p^\beta}\right] \partial_\alpha F = 0, \tag{2.25}$$

since $\partial_\alpha F$ is proportional to ν_α.

It is worth noticing that these jump conditions hold also when there is a distribution of sources on the hypersurface $\sigma^\bullet$. In the special case when there are no such sources, then

$$[p^\beta]\, \partial_\beta F = 0, \tag{2.26}$$

and hence, writing

$$L - p^\beta \frac{\partial L}{\partial p^\beta} = P, \tag{2.27}$$

we have from (2.26)

$$[P]\, \partial_\alpha F = -(p^\beta \partial_\beta F)\left[\frac{\partial L}{\partial p^\alpha}\right]. \tag{2.28}$$

The expression P given by (2.27) is the generalized pressure, as in the case when the Lagrangean takes the form (1.21) P reduces to the usual pressure. The jump-conditions (2.28) then take the usual form also. In fact, let $\vec{n}$ be the space-like unit normal vector to the discontinuity surface, *i.e.*, let $\vec{n} = \operatorname{grad} F/|\operatorname{grad} F|$, where "grad" denotes the spatial gradient. Then

$$G = -\frac{\partial F}{\partial t}\Big/|\operatorname{grad} F|$$

is the speed of displacement of the surface of discontinuity $F(t, x^1, x^2, x^3) = 0$, and equations (2.28) become[12]

$$\varrho(G - v_n)\left[\frac{1}{2}\vec{v}^2 - \frac{\zeta}{\varrho}\right] = v_n[P],$$
$$\varrho(G - v_n)\,[\vec{v}] = \vec{n}\,[P].$$

III. Conservation laws

1. In the calculus of variations the theorems of EMMY NOETHER[13] describe a relationship between the invariance, with respect to given infinitesimal transformations, of the functional considered in a variational principle, and some identities satisfied by the usual Euler expressions of this functional. The essential idea in NOETHER'S theorems can be adapted in the variational principles of hydromechanics considered here, although the hydromechanical variation differs from the usual one.

There are two kinds of Noether theorems. In one, the transformation is supposed to depend on scalar parameters; in the other on functions. Accordingly we shall formulate two theorems corresponding to the hydromechanical variation. In the application of the first theorem, the transformation of the space $\mathfrak{X}_4$ will be taken as the Galilean group. This will restrict the form of the Lagrangean

[12] *Cf.* LICHTENSTEIN [1929], p. 370. As mentioned by LICHTENSTEIN, ZEMPLÉN [1905] was the first to obtained the jump conditions from a variational principle, using, however, Lagrangean variables. See also TAUB [1949], p. 149.

[13] *Cf.* NOETHER [1918], p. 258—276.

and lead to conservation laws of matter, energy, impulse and angular momentum. In the application of the second theorem, the transformation of the space $\mathfrak{X}_4$ will be taken as one for which the hydromechanical variation vanishes identically. This will lead to a generalization of the conservation laws for vorticity. Before stating and proving the corresponding theorems, we give some definitions.

Let

$$W = \int_\tau d\tau\, L(x, p(x)) \tag{3.1}$$

be a functional defined in the space $\mathfrak{P}$, and let τ be any given region contained in $\mathfrak{X}_4$. Consider the infinitesimal transformation of the space $\mathfrak{X}_4$

$$\bar{x}^\alpha = x^\alpha + \Delta x^\alpha, \tag{3.2}$$

where the Δx^α are functions of x, p and their derivatives $\partial_\alpha p$ up to a given order. Consider, further, the infinitesimal transformations of the functions $p(x)$ given by the formula

$$\bar{p}^\alpha = p^\alpha + \partial_\beta (p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta) + \partial_\beta p^\alpha \cdot \Delta x^\beta \tag{3.3}$$

and corresponding to the transformations (3.2) of the x. By means of (3.2) and (3.3) the functional is transformed into

$$\overline{W} = W + \Delta_0 W = \int_{\bar{\tau}} d\tau\, L(\bar{x}, \bar{p}(\bar{x})), \tag{3.4}$$

where $\bar{\tau}$ is the transformed region τ. By writing

$$\delta_0 p^\alpha = \partial_\beta (p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta), \tag{3.5}$$

we have from (3.3), (3.4) and (2.10)

$$\Delta_0 W = \int_\tau d\tau \left\{ \frac{\partial L}{\partial p^\alpha}\, \delta_0 p^\alpha + \partial_\alpha (L \Delta x^\alpha) \right\}. \tag{3.6}$$

If there exists a vector C^α such that

$$\Delta_0 W = \int_\tau d\tau\, \partial_\alpha C^\alpha \tag{3.7}$$

identically in τ, then we call the functional "hydromechanically invariant up to a divergence", or "div-invariant", with respect to the transformations (3.2) and (3.3)[14]. If, in particular, $C^\alpha = 0$, so that $\Delta_0 W = 0$, then W is said to be absolutely invariant with respect to (3.2) and (3.3). Finally, we call the expressions ψ_α given by (2.19) the hydromechanical Euler expressions.

2. Theorem 2. *If the functional W is div-invariant with respect to the transformations* (3.2) *depending essentially on k arbitrary parameters, then exactly k linearly independent linear forms of the Euler expressions ψ_α are divergences.*

The proof is based on the following identity

$$\Delta_0 W = \int_\tau d\tau \{ \partial_\beta (T_\alpha^\beta \Delta x^\alpha) - \psi_\alpha \Delta x^\alpha \}, \tag{3.8}$$

[14] The notion of div-invariance was introduced by E. NOETHER; *cf.* BESSEL-HAGEN [1921], p. 261.

given by (2.17) and (2.18). By the hypothesis of the theorem, the transformations (3.2) are of the form

$$\Delta x^\alpha = \varepsilon^m l_m^\alpha \tag{3.9}$$

where ε^m are infinitesimal scalar parameters, and l_m^α some given functions. Besides,

$$C^\alpha = \varepsilon^m C_m^\alpha, \tag{3.10}$$

where the C_m^α also are given functions. The div-invariance of W is identical in the parameters ε^m; therefore, substituting (3.10) into (3.7) and (3.9) into (3.8), we have from (3.7)

$$\varepsilon^m \int_\tau d\tau\, \partial_\beta C_m^\beta = \varepsilon^m \int_\tau d\tau \{\partial_\beta (T_\alpha^\beta l_m^\alpha) - \psi_\alpha l_m^\alpha\},$$

identically in ε^m and in τ. Hence

$$\psi_\alpha l_m^\alpha = \partial_\beta (T_\alpha^\beta l_m^\alpha - C_m^\beta), \qquad m = 1, 2, \ldots, k, \tag{3.11}$$

which completes the proof.

During the motion we have

$$\psi_\alpha = 0. \tag{3.12}$$

It follows from (3.11) that the functions p^α satisfy the equations

$$\partial_\beta (T_\alpha^\beta l_m^\alpha - C_m^\beta) = 0 \tag{3.13}$$

which are called in physics "conservation laws", usually being written in integral form. Namely, let $\sigma^\bullet$ be any closed hypersurface consisting of two parts $\sigma_{\mathrm{I}}^\bullet$ and $\sigma_{\mathrm{II}}^\bullet$. Integrating (3.13) over the region bounded by $\sigma^\bullet$ and transforming the integral by Gauss' theorem, we obtain an integral form of the conservation laws:

$$\int_{\sigma_{\mathrm{I}}^\bullet} d\tau_\beta (T_\alpha^\beta l_m^\alpha - C_m^\beta) = \int_{\sigma_{\mathrm{II}}^\bullet} d\tau_\beta (T_\alpha^\beta l_m^\alpha - C_m^\beta), \tag{3.14}$$

which asserts that during the motion (3.12) these integrals are conserved. From the general scheme of conservation laws given by (3.13), particular forms can be deduced in accordance with the particular kind of transformation (3.9) assumed. We shall confine ourselves in applying formula (3.13) to the case when (3.9) is the group of Galilean transformation, basic for classical hydromechanics, although the same method can be adapted to other transformations such as the Lorentzian group. Suppose, then, that the action is absolutely invariant with respect to the Galilean transformation

$$\Delta x^\alpha = a^\alpha + a_\beta^\alpha x^\beta, \qquad \alpha, \beta = 0, 1, 2, 3, \tag{3.15}$$

in which the infinitesimal parameters a^α and a_β^α satisfy the conditions

$$a_\beta^0 = a_0^\beta = 0, \qquad a_j^i + a_i^j = 0, \qquad i, j = 1, 2, 3. \tag{3.16}$$

In view of the absolute invariance assumed, in (3.7) we have $C^\alpha = 0$, and, therefore, in (3.6) we have $\Delta_0 W = 0$, identically in the parameters a and in the regions τ. Putting (3.15) and (3.16) into (3.5) to (3.7), we get

$$p^i \frac{\partial L}{\partial p^j} - p^j \frac{\partial L}{\partial p^i} = 0, \qquad \partial_\beta p^\beta \frac{\partial L}{\partial p^\alpha} + \frac{\partial L}{\partial x_{ex}^\alpha} = 0. \tag{3.17}$$

From the first of these equations it follows that the Lagrangean L can depend only on the sum $(p^1)^2+(p^2)^2+(p^3)^2$ and on p^0. Thus the invariance of the action with respect to Galilean transformations restricts the form of the Lagrangean[15]. From the second equation (3.17) the continuity equation $\partial_\beta p^\beta=0$ follows. In fact, $\partial_\beta p^\beta\neq 0$ would imply, by (3.17), $\frac{\partial L}{\partial x^i_{ex}}\Big/\frac{\partial L}{\partial x^j_{ex}}=p^i/p^j$, which is impossible, as the expression on the left-hand side depends on the sum $(p^1)^2+(p^2)^2+(p^3)^2$, while the right-hand side is independent of it.

Other conservation laws are given by (3.13), which yields in this case

$$\partial_\beta T_0^\beta=0,\qquad \partial_\beta T_i^\beta=0,\qquad \partial_\beta M_{ij}^\beta=0 \tag{3.18}$$

with $M_{ij}^\beta=T_i^\beta x_j-T_j^\beta x_i$. Substituting for T_α^β the expression given by (2.18), we get the general scheme of conservation laws for barotropic flows of an inviscid fluid, *viz.*, that of energy, impulse, and angular momentum, respectively. If L takes the particular form (1.21), these laws become the usual ones.

3. Theorem 3. *If the functional W is div-invariant with respect to the transformations*

$$\Delta x^\alpha=\Lambda_s^\alpha[\varphi^s]=A_s^\alpha\varphi^s+A_s^{\alpha\lambda_1}\partial_{\lambda_1}\varphi^s+\cdots+A_s^{\alpha\lambda_1\ldots\lambda_q}\partial_{\lambda_1\ldots\lambda_q}\varphi^s \tag{3.19}$$

depending essentially on r arbitrary functions φ^s, $s=1,2,\ldots,r$, and their derivatives up to a given order q, the coefficients A being given functions, then there exist exactly r linearly independent identities between the Euler expressions ψ_α and their derivatives.

Proof. By the hypothesis, in (3.7) we have

$$C^\alpha=\Gamma_s^\alpha[\varphi^s]=C_s^\alpha\varphi^s+C_s^{\alpha\lambda_1}\partial_{\lambda_1}\varphi^s+\cdots+C_s^{\alpha\lambda_1\ldots\lambda_q}\partial_{\lambda_1\ldots\lambda_q}\varphi^s, \tag{3.20}$$

where the $C_s^{\alpha\cdots}$ are given functions. By substituting (3.19) into (3.6) and (3.8) we have

$$\int_\tau d\tau\,\partial_\alpha(\Gamma_s^\alpha[\varphi^s])=\int_\tau d\tau\{\partial_\beta(T_\alpha^\beta\Lambda_s^\alpha[\varphi^s]-\psi_\alpha\Lambda_s^\alpha[\varphi^s]\} \tag{3.21}$$

identically in the φ^s and in the regions τ. Let $\tilde{\Lambda}_s^\alpha$ be the operator adjoint to the operator Λ_s^α, *i.e.*,

$$\tilde{\Lambda}_s^\alpha[\varphi^s]=A_s^\alpha\varphi^s-\partial_{\lambda_1}(A_s^{\alpha\lambda_1}\varphi^s)+\cdots+(-)^q\partial_{\lambda_1\ldots\lambda_q}(A_s^{\alpha\lambda_1\ldots\lambda_q}\varphi^s). \tag{3.22}$$

Integration of (3.21) by parts yields

$$\begin{aligned}\int_\tau d\tau\,\tilde{\Lambda}_s^\alpha[\psi_\alpha]\,\varphi^s&=\oint_{\sigma^*}d\tau_\beta\left(\Gamma_s^\beta[\varphi^s]+T_\alpha^\beta\Lambda_s^\alpha[\varphi^s]\right)-\\&-\oint_{\sigma^*}d\tau_{\lambda_1}(A_s^{\alpha\lambda_1}\psi_\alpha\varphi^s+A_s^{\alpha\lambda_1\lambda_2}\psi_\alpha\partial_{\lambda_2}\varphi^s+\cdots+A_s^{\alpha\lambda_1\ldots\lambda_q}\psi_\alpha\partial_{\lambda_2\ldots\lambda_q}\varphi^s).\end{aligned} \tag{3.23}$$

As the functions φ^s are arbitrary, we select φ^s which vanish, along with their derivatives up to order $q-1$, on the boundary σ^* of τ. Therefore, from the identity

[15] Apparently, E. & F. COSSERAT [1909] were the first to consider the dependence of the form of the Lagrangean on the invariance properties with respect to a given group of infinitesimal transformations. *Cf. op. cit.*, pp. 8–10, 68–71, 125–127.

(3.23) it follows that

$$\widetilde{\Lambda}_s^{\alpha}[\psi_\alpha] = 0, \tag{3.24}$$

and these are the identities for the Euler expressions ψ_α.

In order to apply Theorem 3 to the hydromechanical case, we have to know the explicit form of the transformation (3.19), which in general depends on the form of the Lagrangean assumed. However, one group of transformations (3.19) can be produced for which the action W is always div-invariant, whatever the form of the Lagrangean may be. This group consists of those Δx^α for which $\delta_0 p^\alpha = 0$ identically, *i.e.*, the hydromechanical variation vanishes, as follows immediately from (3.6) and (3.7) by putting $C^\alpha = L\Delta x^\alpha$. Again, one particular case of this group is evident, *viz.*, $\Delta x^\alpha = p^\alpha \varphi$ for arbitrary functions φ; thus $r=1$, $A^\alpha = p^\alpha$, $q=1$ in (3.19). In this case, Theorem 3, *viz.*, formula (3.24), yields the identity $p^\alpha \psi_\alpha = 0$, showing that the four equations of motion deduced from the variational principle are linearly dependent, as already remarked in Section II. There we saw also that for the usual form of the Lagrangean (1.21) one of the four equations of motion is the energy equation. Now we see that this fact does not depend on the particular from of the Lagrangean.

But the group $\Delta x^\alpha = p^\alpha \varphi$ is only a special case of the more general group given by

Theorem 4. *The hydromechanical variation $\delta_0 p^\alpha = \partial_\beta (p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta)$ vanishes if and only if*

$$\Delta x^\alpha = p^\alpha \varphi + \frac{u_\beta}{p^\nu u_\nu} e^{\alpha\beta\lambda\mu} \partial_\lambda \varphi_\mu, \tag{3.25}$$

where φ is an arbitrary scalar function, u_β is an arbitrary vector, and φ_μ is any vector satisfying the equations

$$p^\lambda (\partial_\lambda \varphi_\mu - \partial_\mu \varphi_\lambda) = 0. \tag{3.26}$$

Proof. First we prove that the condition is sufficient, *i.e.*, that for the Δx^α given by the theorem we have $\delta_0 p^\alpha = 0$. Let us write

$$f^{\alpha\beta} = -f^{\beta\alpha} = e^{\alpha\beta\lambda\mu} \partial_\lambda \varphi_\mu; \tag{3.27}$$

thus

$$\partial_\lambda \varphi_\mu - \partial_\mu \varphi_\lambda = 2 e_{\alpha\beta\lambda\mu} f^{\alpha\beta} \quad \text{and} \quad \partial_\beta f^{\alpha\beta} = 0. \tag{3.28}$$

From (3.28) it follows that

$$p^\lambda e_{\alpha\beta\lambda\mu} f^{\alpha\beta} = 0,$$

which can be written as

$$p^\alpha f^{\beta\gamma} + p^\beta f^{\gamma\alpha} + p^\gamma f^{\alpha\beta} = 0. \tag{3.29}$$

Substituting (3.25) into the expression for $\delta_0 p^\alpha$ yields, in virtue of (3.27), (3.29) and (3.28),

$$\delta_0 p^\alpha = \partial_\beta (p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta) = \partial_\beta \left\{ -\frac{u_\gamma}{p^\nu u_\nu} (p^\beta f^{\gamma\alpha} + p^\alpha f^{\beta\gamma}) \right\}$$

$$= \partial_\beta \left(\frac{p^\gamma u_\gamma}{p^\nu u_\nu} f^{\alpha\beta} \right) = \partial_\beta f^{\alpha\beta} = 0, \qquad \text{q.e.d.}$$

Now we prove that the condition given by the theorem is necessary, *i.e.*, that (3.25), subject to the restriction (3.26), gives all solutions of the equation $\delta_0 p^\alpha = 0$. Let us write

$$p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta = f^{\alpha\beta} = -f^{\beta\alpha}. \tag{3.30}$$

Thus $f^{\alpha\beta}$ satisfies the equations

$$\partial_\alpha f^{\alpha\beta} = 0$$

and

$$p^\alpha f^{\beta\gamma} + p^\beta f^{\gamma\alpha} + p^\gamma f^{\alpha\beta} = 0.$$

The first of these equations implies that

$$f^{\alpha\beta} = e^{\alpha\beta\lambda\mu} \partial_\lambda \varphi_\mu, \tag{3.31}$$

where φ_μ is a vector, and the second one yields (3.26). Let u_γ be any vector. From (3.30) we obtain

$$-p^\alpha u_\gamma \Delta x^\gamma + p^\gamma u_\gamma \Delta x^\alpha = f^{\alpha\beta} u_\beta,$$

and hence, writing

$$\frac{u_\gamma \Delta x^\gamma}{p^\nu u_\nu} = \varphi,$$

we get

$$\Delta x^\alpha = p^\alpha \varphi + \frac{u_\beta}{p^\nu u_\nu} f^{\alpha\gamma} = p^\alpha \varphi + \frac{u_\beta}{p^\nu u_\nu} e^{\alpha\beta\lambda\mu} \partial_\lambda \varphi_\mu,$$

what, together with the equations (3.26) already proved, completes the proof.

For later use we remark that the expression $p^\beta \Delta x^\alpha - p^\alpha \Delta x^\beta$ does not depend on the particular choice of the arbitrary vector u_β. This follows from the formulae (3.30) and (3.31). Therefore, all expressions Δx^α calculated for various u_β by (3.25) differ by a term proportional to the vector p^α only.

4. Now we deduce from Theorems 3 and 4 a generalization of the vorticity theorems of hydromechanics as conservation laws. It would seem to be simplest to express the Δx^α satisfying Theorem 4, *i.e.* as given by (3.25), by means of independent arbitrary functions without any side condition like (3.26), and put them into (3.8). However, such a method is rather cumbersome. Therefore, we shall adapt the idea of Theorem 3 to this case, when instead of (3.19) we have (3.25) with the side condition (3.26).

Thus by (3.6) and (3.8) we have for $\delta_0 p^\alpha = 0$ the identity

$$\int_\tau d\tau\, \psi_\alpha \Delta x^\alpha = \int_\tau d\tau\, \partial_\beta \{(T_\alpha^\beta - \delta_\alpha^\beta L) \Delta x^\alpha\}.$$

We transform the integral on the right-hand side by Gauss' formula into a hyper-surface-integral over the boundary $\sigma^\bullet$ of the region τ considered. On $\sigma^\bullet$ we take $\varphi = \varphi_\mu = 0$ and the vector u_β as normal to $\sigma^\bullet$. Then $\Delta x^\alpha = 0$ on $\sigma^\bullet$, because in the expression (3.25) the derivatives are taken in directions tangential to the hyper-surface $\sigma^\bullet$ only. *E.g.*, in the expression for Δx^1 the derivatives of the component φ_0 are taken in the direction of the vector $u_\gamma e^{1\gamma\lambda 0}$ perpendicular to u_λ, which has been assumed perpendicular to $\sigma^\bullet$. So, by using (2.22), we get

$$\int_\tau d\tau\, \psi_\alpha \Delta x^\alpha = \int_\tau d\tau\, \psi_\alpha \frac{u_\beta}{p^\nu u_\nu} e^{\alpha\beta\lambda\mu} \partial_\lambda \varphi_\mu = 0, \tag{3.32}$$

provided that the side conditions (3.26) hold. These side conditions we take into account by means of Lagrangean multipliers Ω^μ. Thus (3.32) yields

$$\int d\tau \left(\psi_\alpha \frac{u_\beta}{p^\nu u_\nu} e^{\alpha\beta\lambda\mu} + \Omega^\lambda p^\mu - \Omega^\mu p^\lambda\right) \partial_\lambda \varphi_\mu = 0,$$

where, now, the functions φ_μ are arbitrary, provided that they vanish on the boundary $\sigma^\bullet$ of τ. Integrating by parts in the last identity leads to the following conclusion: There exist vectors Ω^μ such that

$$\partial_\lambda \left(\frac{u_\beta}{p^\nu u_\nu} e^{\alpha\beta\lambda\mu} \psi_\alpha\right) = \partial_\lambda (\Omega^\mu p^\lambda - \Omega^\lambda p^\mu). \tag{3.33}$$

These are the identities corresponding to (3.24) in Theorem 3.

Now we shall show that these equations are really generalizations of the vorticity-conservation laws in hydromechanics. To this end we remark first that the functions Ω^μ do not depend on the particular choice of the vectors u_β, because the integral on the left-hand side of (3.32) does not depend on u_β, as follows from the final remark in § 3 and from (2.22). Thus in (3.33) the vector u_β can be taken as completely arbitrary. Secondly, we remark that one of the Lagrangean multiplier Ω^μ can also be chosen arbitrarily, since the side conditions (3.26) are linearly dependent in view of the identity

$$p^\lambda p^\mu (\partial_\lambda \varphi_\mu - \partial_\mu \varphi_\lambda) = 0.$$

Therefore, put $u_0 = 1$, $u_i = 0$, $i = 1, 2, 3$, $\Omega^0 = 0$. The identity (3.33) gives for $\mu = 0$

$$\partial_i (\Omega^i p^0) = 0; \tag{3.34}$$

for $\mu = 1$

$$\partial_2 \left(\frac{\psi_3}{p^0}\right) - \partial_3 \left(\frac{\psi_2}{p^0}\right) = \partial_0 (\Omega^1 p^0) - \partial_2 (\Omega^2 p^1 - \Omega^1 p^2) + \partial_3 (\Omega^1 p^3 - \Omega^3 p^1); \tag{3.35}$$

and analogously for $\mu = 2, 3$.

The meaning of these identities will be clearer if we introduce the usual three-dimensional symbolism, denoting the spatial vectors $(p^0\Omega^1, p^0\Omega^2, p^0\Omega^3)$, (ψ_1, ψ_2, ψ_3) and $\left(\frac{p^1}{p^0}, \frac{p^2}{p^0}, \frac{p^3}{p^0}\right)$ by $\vec{\Omega}, \vec{\psi}$ and $\vec{v}$, respectively. Then (3.34) takes the form

$$\operatorname{div} \vec{\Omega} = 0, \tag{3.36}$$

and (3.35) becomes

$$\operatorname{rot} \frac{\vec{\psi}}{p^0} = \frac{\partial \vec{\Omega}}{\partial t} + \operatorname{rot} (\vec{\Omega} \times \vec{v}), \tag{3.37}$$

or, by use of a known formula,

$$\operatorname{rot} \frac{\vec{\psi}}{p^0} = \frac{\partial \vec{\Omega}}{\partial t} + (\vec{v} \operatorname{grad}) \vec{\Omega} - (\vec{\Omega} \operatorname{grad}) \vec{v} + \vec{\Omega} \operatorname{div} \vec{v}. \tag{3.38}$$

During the motion we have $\vec{\psi} = 0$, and (3.38) reduces to Żorawski's criterion[16],

$$\frac{\partial \vec{\Omega}}{\partial t} + (\vec{v} \operatorname{grad}) \vec{\Omega} - (\vec{\Omega} \operatorname{grad}) \vec{v} + \vec{\Omega} \operatorname{div} \vec{v} = 0, \tag{3.39}$$

for conservation of the vector-lines of the field $\vec{\Omega}$ and the strength of its vector-tubes.

[16] Żorawski [1900]. *Cf.* also Truesdell [1954], pp. 55–57.

It remains only to show that in (3.38) and (3.39) the vector $\vec{\Omega}$ can be taken as a curl of the vector $\partial L/\partial \vec{p}$ with the components $(\partial L/\partial p^1, \partial L/\partial p^2, \partial L/\partial p^3)$, *i.e.*, $\vec{\Omega} = \operatorname{rot} \partial L/\partial \vec{p}$. In fact, calculating the vector $\vec{\psi}/p^0$ occurring in (3.38) by means of (2.19) we get

$$\frac{\psi}{p^0} = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \vec{p}}\right) - \operatorname{grad} \frac{\partial L}{\partial p^0} - \vec{v} \times \operatorname{rot} \frac{\partial L}{\partial \vec{p}},$$

and by using this formula we show that the vector $\operatorname{rot} \dfrac{\partial L}{\partial \vec{p}}$ satisfies (3.38) and (3.39). If the Lagrangean takes the usual form (1.21), then $\partial L/\partial \vec{p} = \vec{v}$, so that $\vec{\Omega}$ becomes the usual curl of the velocity $\vec{\Omega} = \operatorname{rot} \vec{v}$ and the identity (3.38) gives the usual Helmholtz equation.

However, formula (3.39) shows that in addition to $\operatorname{rot} \partial L/\partial \vec{p}$ there exist infinitely many other vectors $\vec{\Omega}$ whose vector-lines and intensities are conserved during the motion $\vec{\psi} = 0$[17]; indeed, it would be sufficient that $\operatorname{rot} \vec{\psi} = 0$.

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[17] This result has been obtained in a different way by Appell [1899]. Another and very short proof of it is due to Truesdell [1954], p. 161.

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A Method of Solution of the Equations of Magnetohydrodynamics

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1. Introduction

The equations of classical magnetohydrodynamics for an infinitely conducting medium fall into two groups. The first group contains the three equations of motion and the equation of continuity, the second group contains equations which involve only the magnetic field and the velocity of the gas. The first group of equations can be thrown into the form of the vanishing of the vectorial divergence of an energy-tensor (see Eqns. (2.14), (2.15) and (2.19) below). This group can be solved in terms of indeterminate functions in a variety of ways, for example, the very general type of solution of equations of this kind found by TRUESDELL (1957) in his study of a problem in elasticity theory could be adapted to the present purpose. Alternatively, the theory already developed by us (BLANKFIELD & McVITTIE 1959, hereafter referred to as BM), which depends on degenerating Einstein's equations of general relativity, may be employed. In this method the indeterminate functions are known to be degenerate forms of the generalized potentials of relativity theory, and one of them, indeed, is the ordinary gravitational potential of Newtonian theory (see the first of Eqn. (2.21) below). But readers who are less interested in the origins of the solutions than in their application to specific physical situations may pass directly to equations (2.21) and verify by direct substitution that the expressions there given do, in fact, satisfy the equation of continuity (2.11) and the three equations of motion (2.14).

In what follows the motion of the gas is regarded as controlled by its pressure-gradient and the magnetic field. The specific illustration which is worked out in detail refers to one-dimensional magnetohydrodynamics in which the dependent variables involve the time and one linear coordinate only. The physical situation is that of a slab of gas, of infinite lateral extent, with parallel plane boundaries. It is shown that in adiabatic motion the slab can expand until it obtains a finite width and an equilibrium configuration.

The same notation is used as in BM: Greek indices range over the values 4, 1, 2, 3, the index 4 being associated with the time, the others with the space coordinates. Latin indices take on the values 1, 2, 3 only and $l\,m\,n$ always denotes a cyclic permutation of 1 2 3. A comma denotes partial differentiation with respect to the coordinate(s) indicated by the subscript(s) following the comma.

2. The Equations of Magnetohydrodynamics

The equations describing the flow of an infinitely conducting non-viscous fluid are (COWLING 1957)

$$\nabla \cdot \boldsymbol{H} = 0, \tag{2.01}$$

$$\nabla_t \boldsymbol{H} + \nabla \times [\boldsymbol{H} \times \boldsymbol{U}] = 0, \tag{2.02}$$

$$\varrho \nabla_t \boldsymbol{U} + \varrho (\boldsymbol{U} \cdot \nabla) \boldsymbol{U} + \nabla p + \mu \boldsymbol{H} \times [\nabla \times \boldsymbol{H}] = 0, \tag{2.03}$$

$$\nabla_t \varrho + \nabla \cdot (\varrho \boldsymbol{U}) = 0, \tag{2.04}$$

where

$$\boldsymbol{J} = \nabla \times \boldsymbol{H}, \tag{2.05}$$

$$\boldsymbol{E} = \mu (\boldsymbol{H} \times \boldsymbol{U}), \tag{2.06}$$

and $\boldsymbol{U}$ is the gas-velocity, ϱ the density, p the pressure, $\boldsymbol{H}$ the magnetic field vector, $\boldsymbol{E}$ the electric field vector, $\boldsymbol{J}$ the current per unit area, μ the permeability, and ∇_t the partial derivative with respect to the time, mks units being employed. For comparison with BM it is necessary to convert these equations to a dimensionless form. Let l, t, V be constants with the physical dimensions of length, time, and gravitational potential (mass/length), respectively, in mks units, and let h be a constant such that

$$\mu h^2 = \frac{V}{t^2}. \tag{2.07}$$

Let the vector $\boldsymbol{H}$ have the three components $h H_i$ where the H_i are dimensionless, and let $\boldsymbol{U}$ have the components $U_i l/t$ where the U_i are dimensionless. Further, a dimensionless density can be obtained from ϱ by multiplying it by l^2/V, and a dimensionless pressure from p by multiplying it by t^2/V. Using ϱ, p to denote these dimensionless variables, (2.01) to (2.04) may now be written

$$\sum_{j=1}^{3} H_{j,j} = 0, \tag{2.08}$$

$$H_{l,4} + (H_l U_m - H_m U_l)_{,m} - (H_n U_l - H_l U_n)_{,n} = 0, \tag{2.09}$$

$$\varrho U_{l,4} + \varrho \sum_{j=1}^{3} U_j U_{l,j} + p_{,l} + H_m (H_{m,l} - H_{l,m}) - H_n (H_{l,n} - H_{n,l}) = 0, \tag{2.10}$$

$$\varrho_{,4} + \sum_{j=1}^{3} (\varrho U_j)_{,j} = 0. \tag{2.11}$$

If equation (2.11) is multiplied by U_l and added to equations (2.10), the following equations are equivalent to (2.10)

$$(\varrho U_l)_{,4} + \sum_{j=1}^{3} (\varrho U_l U_j)_{,j} + p_{,l} + H_m H_{m,l} + H_n H_{n,l} - H_m H_{l,m} - H_n H_{l,n} = 0. \tag{2.12}$$

By adding and subtracting the expression $H_l H_{l,l}$ and adding the expression $-H_l \sum_{j=1}^{3} H_{j,j}$ which, by (2.08), is zero, (2.12) may be written as follows

$$(\varrho U_l)_{,4} + \sum_{j=1}^{3} (\varrho U_l U_j + \delta_{lj} p)_{,j} + \sum_{j=1}^{3} (H_j H_{j,l} - H_j H_{l,j} - H_l H_{j,j}) = 0, \tag{2.13}$$

or

$$(\varrho U_l)_{,4} + \sum_{j=1}^{3} \left[\varrho U_l U_j - H_l H_j + \delta_{lj}\left(p + \frac{H_1^2+H_2^2+H_3^2}{2}\right)\right]_{,j} = 0, \tag{2.14}$$

which replace the equations of motion (2.10).

A formal solution of the equations (2.11) and (2.14) can be arrived at by analogy from the case when the magnetic field is absent. Under this condition, (2.11) is unaltered but (2.14) is

$$(\varrho U_l)_{,4} + \sum_{j=1}^{3} [\varrho U_l U_j + \delta_{lj} p]_{,j} = 0. \tag{2.15}$$

The ten functions ϱ, ϱU_j, $\varrho U_l U_j + \delta_{lj} p$, are the components of the Newtonian energy tensor, $T^{\mu\nu}$, of the fluid and the equations (2.11) and (2.15) are equivalent to the statement that the vectorial divergence of this tensor is zero. There is a corresponding formulation in general relativity of this problem, namely, if $T_g^{\mu\nu}$ is the energy tensor of the fluid then, as is shown in BM, this tensor is related to the metrical and Ricci tensors (in dimensionless variables) by Einstein's equations

$$-\frac{\eta}{\varepsilon^2} T_g^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R, \tag{2.16}$$

where

$$\eta = \frac{8\pi G V}{c^2}, \tag{2.17}$$

$$\varepsilon = \frac{l/t}{c}, \tag{2.18}$$

G is the constant of gravitation and c the velocity of light. It is also known that the vectorial divergence of the right sides of (2.16) vanishes. Therefore, if we calculate the right sides of (2.16) in the Newtonian approximation we obtain expressions for the components of $T^{\mu\nu}$ in terms of the functions involved in the Newtonian forms of $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$. But now the magnetohydrodynamic equations (2.11) and (2.14) differ from the hydrodynamic ones (2.11) and (2.15) only in that the components of the Newtonian energy tensor are

$$\begin{aligned} T^{44} &= \varrho, \\ T^{4l} &= \varrho U_l, \\ T^{lm} &= \varrho U_l U_m - H_l H_m, \\ T^{ll} &= \varrho U_l^2 + p + \tfrac{1}{2}(-H_l^2 + H_m^2 + H_n^2). \end{aligned} \tag{2.19}$$

Thus for the corresponding general relativity problem, a new energy-tensor $T_g^{\mu\nu}$ would be needed in (2.16) whereas the right sides could be retained unaltered. Degenerating to the Newtonian case, the components of the energy-tensor would be given by (2.19) and these would be equated to the degenerate forms of $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$. In BM, the right hand sides of Einstein's equations (2.16) were worked out for the metric

$$\begin{aligned} d\sigma^2 = {} & [1 - \eta(\psi + 2\varepsilon^2 \psi_4)] (d\xi^4)^2 - \\ & - 2\varepsilon^2 \eta \sum_{i=1}^{3} \zeta_i \, d\xi^i \, d\xi^4 - \varepsilon^2 \sum_{i=1}^{3} [1 + \eta(\psi + 2\varepsilon^2 \psi_i)] (d\xi^i)^2, \end{aligned} \tag{2.20}$$

where σ is the dimensionless arc length, ξ^μ dimensionless coordinates, and ψ, ψ_μ, and ζ_i are eight dimensionless indeterminate functions of the ξ^μ. With the energy tensor given by (2.19), the Newtonian approximation to Einstein's equations (2.16) is:

$$\begin{aligned}
\varrho &= -\nabla^2\psi,\\
\varrho U_l &= \psi_{,4l} + \tfrac{1}{2}\Big(\nabla^2\zeta_l - \sum_{i=1}^{3}\zeta_{i,il}\Big),\\
\varrho U_l U_m - H_l H_m &= (\psi_4 - \psi_n)_{,lm} - \tfrac{1}{2}(\zeta_{l,m4} + \zeta_{m,l4}),\\
\varrho U_l^2 + p + \tfrac{1}{2}(-H_l^2 + H_m^2 + H_n^2) &\\
&= -\psi_{,44} + \zeta_{m,4m} + \zeta_{n,4n} + (\psi_n - \psi_4)_{,mm} + (\psi_m - \psi_4)_{,nn}.
\end{aligned} \tag{2.21}$$

These can be regarded as solutions of the equations (2.11) and (2.14) for an arbitrary choice of the eight functions ψ, ψ_μ, ζ_i, as can be verified by direct substitution. However these eight functions must, of course, be determined so that the magnetic field and the velocity satisfy (2.08) and (2.09). It may also turn out that the form of the energy-tensor adopted in (2.21) imposes inter-relations between the eight functions, called consistency relations (McVITTIE 1956).

In one-dimensional motion, all dependent functions involve, by definition, one linear coordinate and the time. Let X be the dimensionless Newtonian linear coordinate, here designated by the index 1, and let T be the dimensionless Newtonian time, designated by the index 4. The equations (2.21) reduce to

$$\varrho = -\psi_{,11}, \tag{2.22}$$

$$\begin{aligned}
\varrho U_1 &= \psi_{,41},\\
\varrho U_2 &= \tfrac{1}{2}\zeta_{2,11},\\
\varrho U_3 &= \tfrac{1}{2}\zeta_{3,11},
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
\varrho U_1 U_2 - H_1 H_2 &= -\tfrac{1}{2}\zeta_{2,14},\\
\varrho U_2 U_3 - H_2 H_3 &= 0,\\
\varrho U_3 U_1 - H_3 H_1 &= -\tfrac{1}{2}\zeta_{3,14},
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
\varrho U_1^2 + p + \tfrac{1}{2}(-H_1^2 + H_2^2 + H_3^2) &= -\psi_{,44},\\
\varrho U_2^2 + p + \tfrac{1}{2}(H_1^2 - H_2^2 + H_3^2) &= -\psi_{,44} + \zeta_{1,41} + (\psi_3 - \psi_4)_{,11},\\
\varrho U_3^2 + p + \tfrac{1}{2}(H_1^2 + H_2^2 - H_3^2) &= -\psi_{,44} + \zeta_{1,41} + (\psi_2 - \psi_4)_{,11}.
\end{aligned} \tag{2.25}$$

Maxwell's equations (2.08) and (2.09) are, in this one-dimensional case

$$\begin{aligned}
H_{1,1} &= 0,\\
H_{1,4} &= 0,\\
H_{2,4} - (H_1 U_2 - H_2 U_1)_{,1} &= 0,\\
H_{3,4} - (H_1 U_3 - H_3 U_1)_{,1} &= 0,
\end{aligned} \tag{2.26}$$

and therefore the component, H_1, of $\boldsymbol{H}$ in the direction of X must be a constant.

3. A Particular One-dimensional Solution

The special case of the solution given by equations (2.22) to (2.26) in which the ζ_i are taken to be zero will now be examined in more detail. The equations (2.22) to (2.25) are now

$$\varrho = -\psi_{,11}, \tag{3.01}$$

$$\begin{aligned} \varrho U_1 &= \psi_{,41}, \\ \varrho U_2 &= 0, \\ \varrho U_3 &= 0, \end{aligned} \tag{3.02}$$

$$\begin{aligned} \varrho U_1 U_2 - H_1 H_2 &= 0, \\ \varrho U_2 U_3 - H_2 H_3 &= 0, \\ \varrho U_3 U_1 - H_3 H_1 &= 0, \end{aligned} \tag{3.03}$$

$$\begin{aligned} \varrho U_1^2 + p + \tfrac{1}{2}(-H_1^2 + H_2^2 + H_3^2) &= -\psi_{,44}, \\ \varrho U_2^2 + p + \tfrac{1}{2}(H_1^2 - H_2^2 + H_3^2) &= -\psi_{,44} + (\psi_3 - \psi_4)_{,11}, \\ \varrho U_3^2 + p + \tfrac{1}{2}(H_1^2 + H_2^2 - H_3^2) &= -\psi_{,44} + (\psi_2 - \psi_4)_{,11}. \end{aligned} \tag{3.04}$$

From equations (3.02) and (3.03), it is easily seen that

$$\begin{gathered} U_1 \neq 0, \quad U_2 = 0, \quad U_3 = 0, \\ H_1 H_2 = H_2 H_3 = H_3 H_1 = 0, \end{gathered}$$

so that the motion is parallel to the direction of **X** and at most one of the magnetic field components is non-zero. The non-trivial case is that in which the magnetic field is not constant, that is, when H_2, the component perpendicular to the velocity, is the non-zero component. In this case, H_2 can be found from the third of equations (2.26) after a substitution has been made for U_1 in terms of ψ from the equations (3.01) and (3.02). Thus

$$H_{2,4} + \left(-\frac{\psi_{,41}}{\psi_{,11}} H_2\right)_{,1} = 0, \tag{3.05}$$

or

$$H_{2,4} - \frac{\psi_{,41}}{\psi_{,11}} H_{2,1} = H_2 \left(\frac{\psi_{,41}}{\psi_{,11}}\right)_{,1}. \tag{3.06}$$

This equation can be solved* with the aid of the equation of continuity (2.11). Dividing equation (3.06) by H_2, and the equation of continuity by ϱ, gives

$$\begin{aligned} \frac{H_{2,4}}{H_2} - \frac{\psi_{,41}}{\psi_{,11}} \cdot \frac{H_{2,1}}{H_2} &= \left(\frac{\psi_{,41}}{\psi_{,11}}\right)_{,1}, \\ \frac{\varrho_{,4}}{\varrho} - \frac{\psi_{,41}}{\psi_{,11}} \cdot \frac{\varrho_{,1}}{\varrho} &= \left(\frac{\psi_{,41}}{\psi_{,11}}\right)_{,1}. \end{aligned}$$

The difference between these is

$$\frac{H_{2,4}}{H_2} - \frac{\varrho_{,4}}{\varrho} - \frac{\psi_{,41}}{\psi_{,11}} \left(\frac{H_{2,1}}{H_2} - \frac{\varrho_{,1}}{\varrho}\right) = 0, \tag{3.07}$$

which has the solution

$$H_2 = \varrho\, \Phi(\psi_{,1}) = -\psi_{,11}\, \Phi(\psi_{,1}), \tag{3.08}$$

* This solution was suggested by A. H. Taub.

where Φ is an arbitrary function of $\psi_{,1}$. Regarding this as the solution for H_2 in terms of ψ, the first of equations (3.04) may then be used to determine p. The last two of equations (3.04) merely serve to determine $(\psi_2-\psi_4)_{,11}$ and $(\psi_3-\psi_4)_{,11}$ when everything else is known. Thus, these last two equations are consistency relations and play no part in finding a solution of equations (3.01) to (3.04)

To show how these steps can be carried out, a particular solution will be obtained for the case of linear waves, by which is meant that the velocity is linear in X, say

$$U_1 = -\frac{\psi_{,41}}{\psi_{,11}} = \frac{F'}{F}\mathsf{X} - F G', \tag{3.09}$$

where F and G are arbitrary functions of T. A prime will denote a derivative of a function with respect to its variable, in the case of F and G therefore, with respect to T. Let

$$Z = \frac{\mathsf{X}}{F} + G; \tag{3.10}$$

then

$$\psi_{,1} = \mathscr{F}'(Z), \tag{3.11}$$

$$\psi = F\mathscr{F}(Z) + H, \tag{3.12}$$

where H is an arbitrary function of T, and $\mathscr{F}$ is an arbitrary function of Z. From (3.08)

$$H_2 = \mathscr{I}(Z)/F, \tag{3.13}$$

where $\mathscr{I}$ is an arbitrary function of Z. The pressure may now be found from the first of equations (3.04) as follows

$$\begin{aligned} p &= -F''\mathscr{F} + \big(ZF'' - (FG)''\big)\mathscr{F}' - H'' - \tfrac{1}{2}\mathscr{I}^2/F^2 \\ &= F''(Z\mathscr{F}' - \mathscr{F}) - (FG)''\mathscr{F}' - H'' - \tfrac{1}{2}\mathscr{I}^2/F^2. \end{aligned} \tag{3.14}$$

It is now necessary to introduce boundary conditions. Suppose that the material occupies an infinite slab bounded by planes perpendicular to the direction of X, and that the pressure and density are taken to be zero at the plane boundaries and also outside the slab. Since

$$\varrho = -\frac{1}{F}\mathscr{F}'', \tag{3.15}$$

let the function $\mathscr{F}''$ vanish for two values of Z, say $Z = \alpha_1$ and $Z = \alpha_2$, so that these points determine the boundaries of the gas cloud. Assume

$$\begin{aligned} &F(T) > 0, \\ &\mathscr{F}''(Z) < 0; \quad \alpha_1 < Z < \alpha_2. \end{aligned} \tag{3.16}$$

Define constants A, B_i, and C_i, for $i = 1, 2$ as follows

$$\begin{aligned} \mathscr{F}'(\alpha_i) &= B_i, \\ \mathscr{F}(\alpha_i) &= C_i, \\ \mathscr{I}(\alpha_i) &= A. \end{aligned} \tag{3.17}$$

The condition that the pressure is to vanish at the boundaries where the density is also zero may be expressed by the following two equations

$$0 = F''(\alpha_i B_i - C_i) - (F\,G)''\,B_i - H'' - \frac{1}{2}\,\frac{A^2}{F^2}, \tag{3.18}$$

the difference between which is

$$F''(\alpha_1 B_1 - \alpha_2 B_2 - C_1 + C_2) = (F\,G)''(B_1 - B_2),$$

or

$$[F(G-B)]'' = 0, \tag{3.19}$$

where

$$B = \frac{\alpha_1 B_1 - \alpha_2 B_2 - C_1 + C_2}{B_1 - B_2}, \tag{3.20}$$

which has the solution

$$G = \frac{\alpha\,T + \beta}{F} + B; \tag{3.21}$$

where α and β are arbirary constants of integration. The two equations (3.18) now yield

$$H'' = F''C - \frac{1}{2}\,\frac{A^2}{F^2}, \tag{3.22}$$

where

$$C = \frac{\alpha_2 B_1 B_2 - \alpha_1 B_1 B_2 + B_2 C_1 - B_1 C_2}{B_1 - B_2}. \tag{3.23}$$

Thus the expression for p is

$$p = F''\{(Z-B)\mathscr{F}' - \mathscr{F} - C\} + \frac{A^2 - \mathscr{I}^2}{2F^2}. \tag{3.24}$$

In the external region, the magnetic field vector must satisfy Maxwell's equations for empty space. The latter reduce, in the one-dimensional case, to a single equation for H_2, namely

$$H_{2,11} = \varepsilon^2 H_{2,44}, \tag{3.25}$$

where $\varepsilon = (l/t)/c$ as in Section 2. The general solution of equation (3.25) is

$$H_2 = f(T + \varepsilon X) + g(T - \varepsilon X), \tag{3.26}$$

where f and g are arbitrary functions. The boundaries occur at $X = (\alpha_1 - G)F$ and $X = (\alpha_2 - G)F$, and H_2 must be continuous there. Hence, using (3.13) and (3.17), the boundary conditions are

$$\begin{aligned} f\big(T + \varepsilon(\alpha_1 - G)F\big) + g\big(T - \varepsilon(\alpha_1 - G)F\big) &= \frac{A}{F}, \\ f\big(T + \varepsilon(\alpha_2 - G)F\big) + g\big(T - \varepsilon(\alpha_2 - G)F\big) &= \frac{A}{F}. \end{aligned} \tag{3.27}$$

The general solutions of these equations for f and g in terms of F will not here be attempted. It is however possible to obtain a solution fairly simply in the symmetric case, that is, the case where the center of the gas cloud remains at rest at the origin. Since the boundaries are at $(\alpha_1 - G)F$ and $(\alpha_2 - G)F$, the

center is at $\left(\frac{\alpha_1+\alpha_2}{2}-G\right)F$. From (3.21), the center is at $\mathsf{X}=\left(\frac{\alpha_1+\alpha_2}{2}-B\right)F-\alpha T-\beta$. The symmetric case is defined by

$$\alpha=\beta=0, \tag{3.28}$$

$$B=\frac{\alpha_1+\alpha_2}{2}. \tag{3.29}$$

Equations (3.28) place a restriction on G, and equation (3.29) places a restriction on $\mathscr{F}$. The restriction on $\mathscr{F}$ is not an impossible one to satisfy, as one could take $\mathscr{F}=\cos Z$ where $B=0$ and $\alpha_2=-\alpha_1=\frac{1}{2}\pi$. In the symmetric case the boundary conditions (3.27) are

$$\begin{aligned} f\left(T+\varepsilon\frac{\alpha_1-\alpha_2}{2}F\right)+g\left(T-\varepsilon\frac{\alpha_1-\alpha_2}{2}F\right)&=\frac{A}{F},\\ f\left(T+\varepsilon\frac{\alpha_2-\alpha_1}{2}F\right)+g\left(T-\varepsilon\frac{\alpha_2-\alpha_1}{2}F\right)&=\frac{A}{F}. \end{aligned} \tag{3.30}$$

The two conditions (3.30) become one condition if g is taken to be the same function as f, thus

$$f\left(T+\varepsilon\frac{\alpha_1-\alpha_2}{2}F\right)+f\left(T-\varepsilon\frac{\alpha_1-\alpha_2}{2}F\right)=\frac{A}{F}, \tag{3.31}$$

and a series for the function f can be found as follows. Let

$$f=\sum_{N=0}^{\infty}\varepsilon^{2N}f_N, \tag{3.32}$$

and expand each of the functions f_N in a Taylor series

$$f_N\left(T\pm\varepsilon\frac{\alpha_1-\alpha_2}{2}F\right)=\sum_{n=0}^{\infty}\left(\pm\varepsilon\frac{\alpha_1-\alpha_2}{2}F(T)\right)^n\frac{1}{n!}f_N^{(n)}(T),$$

where $f_N^{(n)}(T)$ means the n-th derivative of f_N with respect to T. Hence

$$\begin{aligned} \frac{A}{F}&=f\left(T+\varepsilon\frac{\alpha_1-\alpha_2}{2}F\right)+f\left(T-\varepsilon\frac{\alpha_1-\alpha_2}{2}F\right)\\ &=2\sum_{N=0}^{\infty}\sum_{n=0}^{\infty}\varepsilon^{2(N+n)}\left(\frac{\alpha_1-\alpha_2}{2}F(T)\right)^{2n}\frac{1}{(2n)!}f_N^{(2n)}(T), \end{aligned} \tag{3.33}$$

and the functions f_N may be found one at a time by equating powers of ε on both sides of equation (3.33). For ε^0,

$$f_0=\frac{A}{2F},$$

for ε^2,

$$f_1+\frac{1}{2}\left(\frac{\alpha_1-\alpha_2}{2}F\right)^2 f_0''=0,$$

$$f_1=-\frac{A}{4}\left(\frac{\alpha_1-\alpha_2}{2}F\right)^2\left(\frac{1}{F}\right)'',$$

and so on. Thus outside the gas cloud,

$$H_2=f(T+\varepsilon\mathsf{X})+f(T-\varepsilon\mathsf{X}), \tag{3.34}$$

where

$$f=\frac{A}{2F}-\varepsilon^2\frac{A}{4}\left(\frac{\alpha_1-\alpha_2}{2}F\right)^2\left(\frac{1}{F}\right)''+\cdots, \tag{3.35}$$

for the symmetric case.

To proceed further with the solution within the gas cloud, one can impose the additional condition that the rate of change of entropy,

$$\frac{dS}{dT} = \frac{d}{dT}\left[\ln p - \gamma \ln \varrho\right], \tag{3.36}$$

be zero so that the motion is adiabatic. The calculation of this quantity is simplified in the case of linear waves because the derivative following the motion of any function of Z alone is zero. Thus

$$\begin{aligned} \frac{d}{dT}\ln\varrho &= -\frac{F'}{F}, \\ \frac{d}{dT}\ln p &= \frac{F'''\{(Z-B)\mathscr{F}'-\mathscr{F}-C\}-\dfrac{F'}{F^3}(A^2-\mathscr{I}^2)}{F''\{(Z-B)\mathscr{F}'-\mathscr{F}-C\}+\dfrac{1}{2F^2}(A^2-\mathscr{I}^2)}, \end{aligned} \tag{3.37}$$

so that

$$\frac{dS}{dT} = \frac{\left[F'''+\gamma\dfrac{F'}{F}F''\right]\{(Z-B)\mathscr{F}'-\mathscr{F}-C\}-\dfrac{F'}{F^3}\left(1-\dfrac{\gamma}{2}\right)(A^2-\mathscr{I}^2)}{F''\{(Z-B)\mathscr{F}'-\mathscr{F}-C\}+\dfrac{1}{2F^2}(A^2-\mathscr{I}^2)}. \tag{3.38}$$

The adiabatic condition, then, is

$$\left[F'''+\gamma\frac{F'}{F}F''\right]\{(Z-B)\mathscr{F}'-\mathscr{F}-C\} = \frac{F'}{F^3}\left(1-\frac{\gamma}{2}\right)(A^2-\mathscr{I}^2), \tag{3.39}$$

or, assuming that F is not constant,

$$\frac{F^3\left[F'''+\gamma\dfrac{F'}{F}F''\right]}{F'} = \frac{\left(1-\dfrac{\gamma}{2}\right)(A^2-\mathscr{I}^2)}{(Z-B)\mathscr{F}'-\mathscr{F}-C}. \tag{3.40}$$

The left hand side of (3.40) is a function of T alone, while the right hand side is a function of Z alone. Therefore, both must be constant, say C^*. When the right hand side of (3.40) is set equal to C^*, assuming $\gamma \neq 2$, one may solve for $\mathscr{I}^2$ to obtain

$$\mathscr{I}^2 = A^2 - \frac{C^*}{1-\dfrac{\gamma}{2}}\{(Z-B)\mathscr{F}'-\mathscr{F}-C\}. \tag{3.41}$$

When the left hand side of (3.40) is set equal to C^*, the following equation is obtained

$$FF'''+\gamma F'F'' = \frac{C^*F'}{F^2}, \tag{3.42}$$

which can be integrated to give

$$FF''+\frac{\gamma-1}{2}(F')^2 = -\frac{C^*}{F}+A_1, \tag{3.43}$$

if $\gamma \neq 1$. Equation (3.43) can be integrated if $\gamma \neq 2$, after multiplication by the integrating factor, $2F^{\gamma-2}F'$, to give

$$F^{\gamma-1}(F')^2 = -\frac{2}{\gamma-2}C^*F^{\gamma-2}+\frac{2A_1}{\gamma-1}F^{\gamma-1}-A_2. \tag{3.44}$$

The expression for $\mathscr{I}^2$ found in equation (3.41) and the above equations for F may be used to obtain the more compact expression for the pressure

$$p = A_2 \frac{\gamma-1}{2} \frac{1}{F^\gamma} \{(Z-B)\mathscr{F}' - \mathscr{F} - C\}. \tag{3.45}$$

Since the pressure is to be positive in the interval $\alpha_1 < Z < \alpha_2$, it must be shown that the constant A_2 is positive, which can be done as follows. Clearly, the quantity $\frac{1}{2}(\gamma-1)F^{-\gamma}$ is positive since F is positive and $1<\gamma<2$. Let

$$f(Z) = (Z-B)\mathscr{F}' - \mathscr{F} - C.$$

The constants B and C are such that $f(Z)$ vanishes at the boundaries of the gas cloud. Differentiating with respect to Z,

$$f'(Z) = (Z-B)\mathscr{F}''(Z).$$

It has already been assumed in (3.16) that

$$\mathscr{F}''(Z) < 0; \quad \alpha_1 < Z < \alpha_2,$$

so that

$$f'(Z) > 0; \quad \alpha_1 < Z < B,$$
$$f'(Z) < 0; \quad B < Z < \alpha_2.$$

Thus $f(Z)$ is an increasing function of Z for $\alpha_1 < Z < B$, and a decreasing one for $B < Z < \alpha_2$, or $f(Z)$ is positive in the interval $\alpha_1 < Z < \alpha_2$. Hence, in order to have a non-negative pressure, it must be the case that

$$A_2 > 0. \tag{3.46}$$

From the equations of motion (2.10), the force per unit mass including the pressure gradient is seen to be

$$F_l = -\frac{1}{\varrho}[p_{,l} + H_m(H_{m,l} - H_{l,m}) - H_n(H_{l,n} - H_{n,l})], \tag{3.47}$$

and therefore in our case, by

$$\left.\begin{aligned} F_1 &= (Z-B)\left(A_2 \frac{\gamma-1}{2} F^{-\gamma} - \frac{C^*}{2-\gamma} F^{-2}\right), \\ F_2 &= F_3 = 0. \end{aligned}\right\} \tag{3.48}$$

If the expression $A_2 \frac{\gamma-1}{2} F^{-\gamma} - (C^* F^{-2})/(2-\gamma)$ is positive, then the force, F_1, is everywhere directed outward, *i.e.*,

$$F_1 < 0; \quad \alpha_1 \leqq Z < B,$$
$$F_1 > 0; \quad B < Z \leqq \alpha_2.$$

If, on the other hand, the expression $A_2 \frac{\gamma-1}{2} F^{-\gamma} - (C^* F^{-2})/(2-\gamma)$ is negative, then the force is everywhere directed inward, *i.e.*,

$$F_1 > 0; \quad \alpha_1 \leqq Z < B,$$
$$F_1 < 0; \quad B < Z \leqq \alpha_2.$$

Thus, if the constant C^* is negative, the force must be directed outward and the gas cloud must expand. But if

$$C^* > 0, \tag{3.49}$$

then it is possible for the gas cloud to hold together.

For arbitrary γ the equation (3.44) is not soluble in terms of elementary functions. However if $\gamma = \frac{3}{2}$ it can be solved, and this case will now be examined in greater detail. Equation (3.44) is now

$$(F')^2 = \frac{4C^*}{F} - \frac{A_2}{F^{\frac{1}{2}}} + 4A_1. \tag{3.50}$$

In order that F be a real function of T, the right hand side of (3.50) must be non-negative, which will be the case only when

$$F^{-\frac{1}{2}} < \frac{A_2}{8C^*} - \left[\left(\frac{A_2}{8C^*}\right)^2 - \frac{A_1}{C^*}\right]^{\frac{1}{2}},$$

or

$$F^{-\frac{1}{2}} > \frac{A_2}{8C^*} + \left[\left(\frac{A_2}{8C^*}\right)^2 - \frac{A_1}{C^*}\right]^{\frac{1}{2}}.$$

Thus F will always be real if

$$\left(\frac{A_2}{8C^*}\right)^2 - \frac{A_1}{C^*} \leqq 0,$$

or

$$A_1 \geqq C^* \left(\frac{A_2}{8C^*}\right)^2.$$

In particular, consider the case where

$$A_1 = C^* \left(\frac{A_2}{8C^*}\right)^2. \tag{3.51}$$

Equation (3.50) becomes, in this case,

$$F' = \left[\frac{4C^*}{F} - \frac{A_2}{F^{\frac{1}{2}}} + 4C^*\left(\frac{A_2}{8C^*}\right)^2\right]^{\frac{1}{2}} = 2(C^*)^{\frac{1}{2}}\left[F^{-\frac{1}{2}} - \frac{A_2}{8C^*}\right], \tag{3.52}$$

which has the solution

$$T + A_3 = -\frac{1}{(C^*)^{\frac{1}{2}}}\left[\left(\frac{8C^*}{A_2}\right)^3 \ln\left(1 - \frac{A_2}{8C^*}F^{\frac{1}{2}}\right) + \frac{8C^*}{A_2}\left(\frac{1}{2}F + \frac{8C^*}{A_2}F^{\frac{1}{2}}\right)\right]. \tag{3.53}$$

As T tends to $+\infty$, F tends to the finite value $\left(\frac{8C^*}{A_2}\right)^2$, from below. Since the width of the gas cloud is directly proportional to F, the cloud expands, but only to a finite width. Thus a solution of the equations of magnetohydrodynamics has been obtained describing the motion of a gas cloud with infinite plane faces normal to the direction of motion, the forces being the pressure gradient and a magnetic force parallel to the faces. The pressure and density vanish on the plane faces, the motion is adiabatic and the ratio of specific heats has the value $\frac{3}{2}$. The gas cloud expands, but not beyond a certain finite width. Ultimately, the binding effect of the magnetic field balances the disruptive effect of the pressure gradient, and the cloud approaches an equilibrium configuration.

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On the Convergence of the Rayleigh Quotient Iteration for the Computation of the Characteristic Roots and Vectors. II

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In what follows I add some observations to Part I of this paper*, which the reader is presumed to have at hand. The enumeration of sections and formulae continues that of Part I.

18. In the paper by Crandall referred to in § 17 the cubic character of the convergence is proved in the following sense for the *sequence of vectors* ξ_k obtained by the rule of § 9:

If $\xi_k \to \eta$, *and if for a sequence of numbers* ε_k *with* $\varepsilon_k \to 0$,

$$\xi_k - \eta = O(\varepsilon_k) \qquad (k \to 0), \tag{58}$$

then

$$\xi_{k+1} - \eta = O(\varepsilon_k^3) \qquad (k \to 0). \tag{59}$$

However, from this result it *does not follow* that $\lambda_{k+1} - \sigma = O\left((\lambda_k - \sigma)^3\right)$, and still less that the asymptotic formula (46) holds.

On the other hand, the exact relation (46) presents not only theoretical but also practical interest because by use of this relation the convergence can be hastened still more without much additional work.

19. From (46) follows

$$\begin{gathered} \frac{\lambda_k - \lambda_{k+1}}{\lambda_k - \sigma} = \frac{\lambda_k - \sigma - (\lambda_{k+1} - \sigma)}{\lambda_k - \sigma} = 1 + O\left((\lambda_k - \sigma)^2\right), \\ \lambda_k - \sigma \sim \lambda_k - \lambda_{k+1}, \quad \lambda_{k+1} - \sigma \sim \lambda_{k-1} - \lambda_{k+1}. \end{gathered} \tag{60}$$

Therefore, by applying (46) again and replacing k by $k-1$ we see that

$$\frac{\lambda_k - \lambda_{k+1}}{(\lambda_{k-1} - \lambda_{k+1})^3} \to \gamma,$$

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* Arch. Rational Mech. Anal. **1**, 233–241 (1958).

and by comparing this with (46) and again using (60), since $\gamma \neq 0$,

$$\frac{\lambda_{k+1}-\sigma}{(\lambda_k-\lambda_{k+1})^3} \sim \frac{\lambda_k-\lambda_{k+1}}{(\lambda_{k-1}-\lambda_{k+1})^3},$$

$$\sigma-\lambda_{k+1} \sim \frac{(\lambda_k-\lambda_{k+1})^4}{(\lambda_{k+1}-\lambda_{k-1})^3}.$$

We see that replacing λ_{k+1} by

(61) $$\lambda'_{k+1} = \lambda_{k+1} + \frac{(\lambda_{k+1}-\lambda_k)^4}{(\lambda_{k+1}-\lambda_{k-1})^3},$$

gives an even better approximation to σ than λ_{k+1}. Using this remark systematically, we shall apply (61) *at every second step*. Starting with $\xi_0=\eta$ we compute

$$\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2$$

by (28) and (29) and replace λ_2 by λ'_2, using (61). Then we start with λ'_2 and ξ_2 and proceed in the same manner. The general double step starts with λ'_{2m-2} and ξ_{2m-2}. Then we compute ξ_{2m-1}, λ_{2m-1}, ξ_{2m}, λ_{2m} using (28) and (29) and then again replace λ_{2m} by λ'_{2m} using (61). As a result of a discussion published elsewhere*, it is expected that the gain in precision due to the above improvement is about 11%.

20. In computing practice the cubic character of the convergence appears only *at the second step*. At the first step the error is approximately only squared. Indeed, we have from (13) for $k=0$

$$\frac{\lambda_1-\sigma}{(\lambda_0-\sigma)^2} = \frac{\sum' \frac{\sigma_\mu-\sigma}{(\sigma_\mu-\lambda_0)^2} p_\mu}{p+\sum'\left(\frac{\sigma-\lambda_0}{\sigma_\mu-\lambda_0}\right)^3 p_\mu}.$$

But from this it follows, if $\lambda_0 \to \sigma$, that

(62) $$\frac{\lambda_1-\sigma}{(\lambda_0-\sigma)^2} \to \frac{1}{p}\sum' \frac{p_\mu}{\sigma_\mu-\sigma}.$$

In order to discuss in an analogous manner the order of $\lambda_2-\sigma$, we rewrite (34) for $k=1$, making use of (33):

$$\lambda_2-\sigma = \frac{\sum\limits_{\mu=1}^{m}{}' \frac{(\sigma_\mu-\sigma)}{M^2_{\mu,1}} p_\mu}{\frac{p}{(\sigma-\lambda_0)^2(\sigma-\lambda_1)^2} + \sum\limits_{\mu=1}^{m}{}' \frac{p_\mu}{M^2_{\mu,1}}},$$

or

$$\frac{\lambda_2-\sigma}{(\lambda_0-\sigma)^2(\lambda_1-\sigma)^2} = \frac{\sum\limits_{\mu=1}^{m}{}' \frac{\sigma_\mu-\sigma}{M^2_{\mu,1}} p_\mu}{p+(\sigma-\lambda_0)^2(\sigma-\lambda_1)^2 \sum\limits_{\mu=1}^{m}{}' \frac{p_\mu}{(\sigma_\mu-\lambda_0)^2(\sigma_\mu-\lambda_1)^2}}.$$

Now if we let λ_0 approach σ, we see that $\lambda_1 \to \sigma$, $\lambda_2 \to \sigma$, and it follows that

$$\frac{\lambda_2-\sigma}{(\lambda_0-\sigma)^2(\lambda_1-\sigma)^2} \to \frac{1}{p}\sum_{\mu=1}^{m}{}' \frac{p_\mu}{(\sigma_\mu-\sigma)^3}.$$

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* Ostrowski, A. M.: A method of speeding up iterations with super-linear convergence. J. Mathematics and Mechanics 7, 117—120 (1958).

Dividing this by (62) we obtain finally

$$\frac{\lambda_2-\sigma}{(\lambda_1-\sigma)^3} \to \frac{\sum\limits_{\mu=1}^{m}{}' \frac{p_\mu}{(\sigma_\mu-\sigma)^3}}{\sum\limits_{\mu=1}^{m}{}' \frac{p_\mu}{\sigma_\mu-\sigma}}, \tag{63}$$

provided that the denominator on the right-hand side does not vanish.

Observe that the limit on the right-hand side, valid for $\lambda_0 \to \sigma$, is in general different from the limit γ in (46), valid for $k \to \infty$.

21. If, in the above iteration, the sequence λ_k converges to an eigenvalue σ, we can obtain also a corresponding eigenvector as the limit of the vector sequence $\xi_k/|\xi_k|$. Indeed, it follows from (30), (42) and (47) that if $y_{i_1}, \ldots, y_{i_l}$ are the components of η corresponding to the eigenvalue σ, then the sequence $\xi_k/|\xi_k|$ tends to a vector ζ. The components of ζ which correspond to the eigenvalues distinct from σ vanish, while the components with indices $i_1, \ldots, i_l$ are given by

$$\frac{y_{i_1}}{\sqrt{y_{i_1}^2+\cdots+y_{i_l}^2}}, \ldots, \frac{y_{i_l}}{\sqrt{y_{i_1}^2+\cdots+y_{i_l}^2}}.$$

After a certain number of eigenvalues and corresponding independent eigenvectors have been obtained by the above process, the method can be applied in such a way that in any case it gives either a new eigenvalue or at least a new eigenvector linearly independent of the eigenvectors already found. Suppose, indeed, that a complete linearly independent set of unimodular eigenvectors already found is given by $\zeta_1, \ldots, \zeta_l$. Then choose a starting vector η orthogonal to all ζ_λ, *i.e.*, such that

$$(\zeta_\lambda, \eta) = 0 \qquad (\lambda = 1, \ldots, l). \tag{64}$$

Then I say that *all vectors* ξ_k deduced by the above process from η *are orthogonal to each of the* ζ_λ. Indeed, introducing normal coordinates as in § 2, we make $\zeta_1, \ldots, \zeta_l$ the coordinate unit vectors corresponding to the indices $1, \ldots, l$. But then the orthogonality relation (64) reduces to

$$y_1 = \cdots = y_l = 0,$$

and by (30) it follows that for each k

$$x_1^{(k)} = \cdots = x_l^{(k)} = 0;$$

that is, ξ_k is orthogonal to each ζ_λ. Therefore the limiting value of $\xi_k/|\xi_k|$ is also orthogonal to all ζ_λ and is independent of them. Thus our procedure started in this way gives us a new eigenvector independent of those already obtained.

It is hardly necessary to mention that after $n-1$ linearly independent eigenvectors have been obtained, the n-th eigenvector is determined by the $n-1$ relation (64), and the corresponding eigenvalue is obtained at once as the value of the Rayleigh quotient.

Further, it is easily seen to be sufficient that the relations (64) are satisfied with a certain approximation in order to be sure that the new eigenvector will not be a linear combination of the eigenvectors $\xi_1, \ldots, \xi_n$.

22. In applying the iteration rule (28), (29), the largest amount of computation is involved in the solution of the linear system with the matrix $A-\lambda_k E$, since this solution implies each time in the general case a multiple of n^3 elementary operations. It can be worthwhile therefore to reduce A by a preliminary transformation to a tridiagonal form, since for a system with a tridiagonal matrix the number of elementary operations implied in the solution of the corresponding linear system is only a multiple of n^2, and the tridiagonal form is preserved in all the matrices $A-\lambda_k E$. On the other hand, as has been pointed out by W. GIVENS, the reduction of an Hermitian matrix to a tridiagonal form can be carried out by a *finite* number of "binary rotations"*.

However, the reduction to tridiagonal form would have to be carried out from the beginning with the greatest precision that may be required throughout the computation. Further, in this way we obtain directly only the approximations to the *eigenvalues* of the original matrix, since it probably would not be worthwhile to obtain from the eigenvectors of the transformed matrix those of the original matrix by applying again the whole sequence of binary rotations. And if we then obtain the corresponding eigenvectors of the original matrix A by applying the rule (5), we still have to solve the corresponding system of linear equations with an ill-conditioned general matrix.

23. The eigenproblems in the case of a finite number of degrees of freedom usually present themselves not in the form

$$A\xi=\lambda\xi \tag{65}$$

but in the more general form

$$A\xi=\lambda B\xi. \tag{66}$$

In the form (66) B is a positive definite symmetric matrix while A is, as usually assumed, a real symmetric matrix.

It is easy to deduce from our results the corresponding rules and theorems in the more general case (66). Observe that a positive symmetric matrix B can be written as a square of a symmetric positive matrix Δ

$$B=\Delta^2. \tag{67}$$

Introducing this expression of B in (66) and multiplying on the left by Δ^{-1}, we obtain

$$\Delta^{-1}A\,\Delta^{-1}(\Delta\xi)=\lambda(\Delta\xi);$$

therefore, putting

$$\Delta^{-1}A\,\Delta^{-1}=A_1,\qquad \Delta\xi=\xi_1, \tag{68}$$

$$A_1\xi_1=\lambda\xi_1, \tag{69}$$

we see that any eigenvalue λ of (66) is also an eigenvalue of (69) and the corresponding eigenvector is obtained by (68)**.

* A finite reduction to a tridiagonal form was contained implicitly in some previous papers by C. LANCZOS but was not worked out explicitly.

** *Cf.* CHERUBINO: Su un'equazione della teoria delle vibrazioni. Boll. Un. mat. Ital., Ser. III, **11**, 133–136 (1956).

If for an arbitrary vector ξ, the Rayleigh quotient is now formed relative to the matrix A_1, it can be written in terms of $\xi = \Delta^{-1}\xi_1$, A and B by use of (69):

$$\frac{\xi_1' A_1 \xi_1'}{\xi_1' \xi} = \frac{\xi' A \xi'}{\xi' B \xi} = \frac{Q_A(\xi)}{Q_B(\xi)}.$$

On the other hand if we derive from ξ_1 a vector $\xi_1^{(1)}$ by

$$(A_1 - \lambda_0 E)\,\xi_1^{(1)} = \xi_1,$$

this is reduced by (69) to

$$(A - \lambda_0 B)\,\xi^{(1)} = B\,\xi.$$

We see that our iteration rule (28), (29) in the case of (66) is transformed into the rule

(70) $$\xi^{(k)} = (A - \lambda_k B)^{-1} B\,\xi^{(k-1)},$$

(71) $$\lambda_{k+1} = \frac{Q_A(\xi_k)}{Q_B(\xi_k)},$$

and our main result (46) is therefore valid in the case (66) too*.

24. *Examples***.

1. The matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ has 2 and 7 as eigenvalues. We begin with the vector $\xi_0 = (1, 0)$, obtain by (28) $\lambda_0 = 2$, thence compute $\xi_1, \lambda_1, \ldots, \xi_3, \lambda_3$, and finally apply to $\lambda_1, \lambda_2, \lambda_3$ the improvement contained in the formula (61). The results are given in the following table:

i	λ_i		$x_1^{(i)}$	$x_2^{(i)}$
0	3		1	0
1	2.076923		1.5	-1
2	$2.0^4 19073413568826$		$2.0^2 9803921568627451$	-1
3, 3′	$2.0^{15}277$	$1.9^{16}36$	$1.9^7 62747097570$	-1

The convergence is particularly fast, since we have in this case $\gamma = \frac{1}{(\sigma_1 - \sigma_2)^2} = \frac{1}{25} = 0.04$. The values of the quotients $\frac{\lambda_{k+1} - \sigma}{(\lambda_k - \sigma)^3}$, $(k = 0, 1, 2)$, $\frac{\lambda_3' - \sigma}{\lambda_3 - \sigma}$ are 0.0769; 0.0419; 0.0399; 0.05054.

2. The matrix

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{pmatrix}$$

has $\sigma_1 = \sigma_2 = 1$ as a double eigenvalue, and $\sigma_3 = 15$ as the third eigenvalue. Starting with the vector $\xi_0 = (1, 0, 0)$, we obtain the values of $\lambda_0, \xi_1, \lambda_1, \xi_2, \lambda_2, \xi_3, \lambda_3$

* The rule (70), (71) is discussed directly in CRANDALL's paper cited above.

** For the computation of these examples I am greatly indebted to Mrs. IDA RHODES, National Bureau of Standards, Washington D.C.

and derive again from λ_1, λ_2, λ_3 the value of λ_3 by (61). The results are contained in the table:

i	λ_i	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$
0	2	1	0	0
1	$1.0^{2}63694267515923566878980891 71974$	-12	2	3
2	$1.0^{8}13201943358809031686025 16$	$-13.0^{3}4901960784313 72$	2	3
3, 3′	$1.0^{28}11739$ | $0.9^{31}83$	$-12.9^{13}53776$	2	3

The value of $\frac{\lambda_3'-\sigma}{\lambda_3-\sigma}$ and those of $\frac{\lambda_{k+1}-\sigma}{(\lambda_k-\sigma)^3}$ ($k=0, 1, 2$) are in this case 0.001 448, 0.006 369, 0.005 109, 0.005 1017 while $\gamma = \frac{1}{196}$ is 0.005 102 04.

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A Short Proof of a Boundedness Theorem for Linear Differential Systems with Periodic Coefficients

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Introduction. In the last few years a number of boundedness theorems for homogeneous, linear, differential systems with periodic coefficients and containing a small parameter have been proved by L. CESARI, the author and others, by using a method of successive approximations first devised by CESARI. More recently, M. GOLOMB [6] has proved similar theorems by using another method of successive approximations devised by WEBER (see [2], in particular § 4, no. 5, for general references). The aim of the present paper is to show that an elementary argument leads to a new general theorem of boundedness which comprehends as a particular case some (though not all) of the statements previously proved by the method of CESARI and by the method of WEBER.

Consider the system*:

$$\begin{aligned} u_j'' + A_j u_j &= \sum_{k=1}^{2} [\Phi_{jk}(t,\lambda)\, u_k + \Psi_{jk}(t,\lambda)\, u_k'] + \Psi_{j3}(t,\lambda)\, u_3, \qquad j=1,2, \\ u_3' &= \sum_{k=1}^{2} [\Phi_{3k}(t,\lambda)\, u_k + \Psi_{3k}(t,\lambda)\, u_k'] + \Psi_{33}(t,\lambda)\, u_3, \end{aligned} \tag{1}$$

where λ is a real parameter, u_3 is a scalar, u_1 is an m-dimensional vector, $0 \leqq m \leqq n$, u_2 is an $n-m$-dimensional vector, $n \geqq m$, and each $\Phi_{jk}(t,\lambda)$, $\Psi_{jk}(t,\lambda)$ is a matrix whose dimension is determined by (1) and the elements of these matrices are real-valued functions of the real variable t, periodic in t of period $T = 2\pi/\omega$, L-integrable in $[0, T]$, and are continuous functions of λ at $\lambda = 0$ for almost all t in $[0, T]$. If $p(t,\lambda)$ is any element of these matrices, then we assume that $p(t, 0) = 0$ and there exists a function $q(t)$, L-integrable in $[0, T]$, and a $\lambda_0 > 0$ such that $|p(t,\lambda)| < q(t)$ almost everywhere in $[0, T]$, $0 \leqq |\lambda| \leqq \lambda_0$. The class of functions $p(t,\lambda)$ which satisfy these properties will be called $A(T)$. Furthermore, $A_1 = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, $A_2 = \operatorname{diag}(\sigma_{m+1}^2, \ldots, \sigma_n^2)$ where $\sigma_1, \ldots, \sigma_n$ are positive real numbers with $\sigma_j \pm \sigma_k \neq r\omega$, $2\sigma_j \neq r\omega$, $\sigma_j \neq r\omega$, $j \neq k$, $j, k = 1, 2, \ldots, n$, $r = 0, 1, 2, \ldots$.

We shall say that a function $f(t)$ is *even with respect to* γ if $f(t+\gamma) = f(-t)$ and that it is *odd with respect to* γ if $f(t+\gamma) = -f(-t)$.

* Throughout the present paper, two functions are said to be equal if they are equal except for a set of Lebesgue measure zero.

As it is immediately seen, these two concepts coincide with the usual evenness and oddness when the translatation $t=t_1-\frac{1}{2}\gamma$ is made. For real functions $f(t)$ periodic of period $T=2\pi/\omega$, L-integrable in $[0, T]$, with Fourier coefficients $a_0, a_k, b_k, k=1, 2, \ldots$, the following cases are of interest: $f(t)$ is even with respect to (a) $\gamma=0$ if $f(t)=f(-t)$, (b) $\gamma=\frac{1}{2}T$, if $a_{2k-1}=b_{2k}=0$, $k=1, 2, \ldots$, (c) $\gamma=T/4k$ if $f(t)=a(\cos k\omega t+\sin k\omega t)$; $f(t)$ is odd with respect to (a') $\gamma=0$, if $f(t)=-f(-t)$, (b') $\gamma=T/2$ if $a_{2k}=b_{2k-1}=0$, $k=1, 2, \ldots$, (c') $\gamma=T/4k$ if $f(t)=a\cos 2k\omega t+b\sin 4k\omega t$. We prove the following:

Theorem 1. *If the matrices Φ_{jk}, Ψ_{jk} of system (1) are such that $\Phi_{jk}(-t)=(-1)^{k+j}\Phi_{jk}(t+\gamma)$, $\Psi_{jk}(-t)=(-1)^{k+j+1}\Psi_{jk}(t+\gamma)$ for some real number γ, then there exists a $\lambda_1>0$ such that all of the absolutely continuous (AC) solutions of (1) are bounded for $-\infty<t<+\infty$, $0\leq|\lambda|<\lambda_1$.*

In case all of the coefficients in the last equation of system (1) are zero and the coefficients of u_3 are zero in the first n equations of (1), then Theorem 1 is a boundedness theorem for systems of second order differential equations. For this case, Theorem 1 for $\gamma=0$, $m=n$ and all the matrices $\Psi_{jk}=0$ was first proved by L. CESARI [*3*] and was later extended by R. A. GAMBILL [*5*] to the case where $\Psi_{jk}\neq 0$. For $\gamma=0$, $m\neq n$, Theorem 1 was first proved by the author [*8*] and later, for all $\Psi_{jk}=0$, by M. GOLOMB [*6*]. Then the elements of the matrices Φ_{jk}, Ψ_{jk} are either even functions of t or odd functions of t.

The elementary argument used in this paper for the proof of Theorem 1 seems to be new, though some elements of it can be traced to A. LYAPUNOV [*10*, p. 409] for systems of order n and to L. CESARI & J. K. HALE [*4*] for systems of order $n=2$. On the other hand, by using the method of CESARI, the author [*9*] has been able to obtain a boundedness theorem which includes Theorem 1 for systems of type (1) where more than one characteristic root for $\lambda=0$ may be zero. The emphasis in this paper is to show that many results concerning the qualitative behavior of the solutions of systems of type (1) can be obtained without using successive approximations.

Proof. In this section we prove Theorem 1, but we begin by discussing the properties of a system of differential equations which is slightly more general than system (1). Consider

$$z'=Az+C(t,\lambda)z \tag{2}$$

where λ is a real parameter, $z=(z_1, \ldots, z_{2n+1})$, A is a $(2n+1)\times(2n+1)$ matrix whose elements are real constants and $C(t,\lambda)$ is a $(2n+1)\times(2n+1)$ matrix whose elements belong to the class $A(T)$. Furthermore, we shall suppose that the characteristic roots v_k, $k=1, 2, \ldots, 2n+1$, of A are $v_{2j-1}=i\sigma_j$, $v_{2j}=-i\sigma_j$, $v_{2n+1}=0$, where $\sigma_1, \ldots, \sigma_n$ are positive, real numbers such that $v_j\not\equiv v_k \pmod{\omega i}$, $j\neq k, j, k=1, 2, \ldots, 2n+1$.

Let

$$d=\min_{\substack{j,k=1,2,\ldots,2n+1\\ m=0,1,\ldots,|j-k|+m>0}}|im\omega-(v_j-v_k)|,$$

and let ε be any real number such that $0<\varepsilon<\frac{1}{2}$, and consider, in the complex plane, circles $C_1, \ldots, C_{2n+1}$ with radii εd such that C_j has center at v_j, $j=1, 2, \ldots, 2n+1$. If $\mu_1, \ldots, \mu_{2n+1}$ are any complex numbers such that $\mu_j\in C_j$,

$j=1, 2, \ldots, 2n+1$, then

$$\min_{\substack{j,k=1,2,\ldots,2n+1 \\ m=0,1,2,\ldots,|j-k|+m>0}} |\operatorname{im}\omega - (\mu_j - \mu_k)| \geqq d - 2\varepsilon d = d(1-2\varepsilon) > 0. \qquad (3)$$

Let $\tau_j = \tau_j(\lambda)$, $j=1, 2, \ldots, 2n+1$, denote the characteristic exponents of system (2). The numbers τ_j, of course, are only determined up to a multiple of ωi, but we may assume without loss of generality that $\tau_{2j-1}(0) = i\sigma_j$, $\tau_{2j}(0) = -i\sigma_j$, $j=1, 2, \ldots, n$, $\tau_{2n+1}(0)=0$. Furthermore, exactly as in [*4*], it can be shown that the above conditions on the matrix $C(t, \lambda)$ imply that these characteristic exponents for system (2) are continuous functions of λ at $\lambda=0$ and there exists a $\lambda_0'>0$ such that for $0 \leqq |\lambda| \leqq \lambda_0'$, $\tau_{2j-1}(\lambda) = \bar{\tau}_{2j}(\lambda)$, $j=1, 2, \ldots, n$, and τ_{2n+1} is real. Also, since $\tau_j(\lambda)$ is continuous at $\lambda=0$, there exists a $\lambda_0''>0$ such that $\tau_j(\lambda) \in C_j$ for $0 \leqq |\lambda| \leqq \lambda_0''$, which implies from (3) that

$$\tau_j(\lambda) \not\equiv \tau_k(\lambda) \,(\operatorname{mod} \omega i), \qquad j \neq k,\ j, k = 1, 2, \ldots, 2n+1. \qquad (4)$$

Therefore, we have the following:

Theorem 2. *For $0 \leqq |\lambda| \leqq \lambda_1$, $\lambda_1 = \min(\lambda_0, \lambda_0', \lambda_0'')$, the characteristic exponents $\tau_j(\lambda)$, $j=1, 2, \ldots, 2n+1$, of system (2) have the following properties:*

(i) $\tau_{2j-1}(\lambda) = \bar{\tau}_{2j}(\lambda)$, $\quad j=1, 2, \ldots, n$, $\tau_{2n+1}(\lambda) = \bar{\tau}_{2n+1}(\lambda)$,

(ii) $\tau_j(\lambda) \not\equiv \tau_k(\lambda) \,(\operatorname{mod} \omega i)$, $\quad j \neq k, j, k = 1, 2, \ldots, 2n+1$.

If $\beta_j = e^{\tau_j T}$, $j=1, 2, \ldots, 2n+1$, and $0 \leqq |\lambda| \leqq \lambda_1$, then, it follows from Theorem 2 that the β_j are distinct complex numbers. It is known that, when all of the characteristic multipliers β_j are distinct, there is a fundamental system of AC solutions of (2) of the form

$$\{z_{jk} = e^{\tau_j t} p_{jk}(t),\ k=1, 2, \ldots, 2n+1\}, \qquad j = 1, 2, \ldots, 2n+1, \qquad (5)$$

where each $p_{jk}(t) = p_{jk}(t+T)$.

Theorem 3. *Let $z(t)$ be any solution of (2); let P be a real constant nonsingular matrix which is independent of λ; and let $(*)$ denote the differential equation for $w(t) = Pz(-t)$ which is obtained from (2) by replacing t by $-t$ and applying the transformation P. If there exists a real constant nonsingular matrix Q which is independent of λ and such that $w(t) = Qz(t)$ also satisfies $(*)$, then all of the AC solutions of (2) are bounded for $-\infty < t < +\infty$, $0 \leqq |\lambda| < \lambda_1$, where λ_1 is defined above.*

Proof. For $0 \leqq |\lambda| \leqq \lambda_1$, $Z(t) = \{z_{jk}(t), j, k = 1, 2, \ldots, 2n+1\}$ defined by (5) represents a fundamental system of AC solutions of (2). It is clear that $PZ(-t)$ is a fundamental system of AC solutions of $(*)$. If $z(t)$ is any solution of (2), then there exist a constant vector a such that

$$z(t) = Za.$$

We choose $z(t)$ so that each of the components of the vector a are different from zero. On the other hand, since $Qz(t)$ satisfies $(*)$, there exists a constant vector b such that

$$Qz(t) = PZ(-t)\,b.$$

But, the previous formula together with this one implies that

$$Q Z(t)\, a = P Z(-t)\, b.$$

However, from Lemma (2.ii) of [7], this formula implies that there exists, for every $j=1, 2, \ldots, 2n+1$, an $l(j)$ such that $-\tau_j \equiv \tau_l \pmod{\omega i}$; or, if $\Re(a)$ and $\Im(a)$ denote the real and imaginary parts of a, respectively,

$$-\Re(\tau_j) \equiv \Re(\tau_l) \pmod{\omega i},$$
$$-\Im(\tau_j) \equiv \Im(\tau_l) \pmod{\omega i}.$$

From Theorem 2, τ_l must be the complex conjugate of τ_j and, thus, $R(\tau_j)=0$, $j=1, 2, \ldots, 2n+1$. Finally, from (5), all of the AC solutions of (2) are bounded for $-\infty<t<\infty$, $0\leqq|\lambda|<\lambda_1$ and the theorem is proved.

If the vectors u_1, u_2 in (1) are given by $u_1=(u_{11}, \ldots, u_{1m})$, $u_2=(u_{21}, \ldots, u_{2,n-m})$, then the transformation of variables

$$\begin{aligned} &z_j = u_{ij}, && z_{n+j} = u'_{1j}, && j=1, 2, \ldots, m, \\ &z_{m+j} = u_{2j}, && z_{n+m+j} = u'_{2j}, && j = 1, 2, \ldots, n-m, \\ &z_{2n+1} = u_3 \end{aligned} \tag{6}$$

leads to the equivalent system of $2n+1$ first order equations

$$z' = A z + C(t, \lambda)\, z, \tag{7}$$

where

$$\begin{aligned} &A = \begin{pmatrix} O_{n,n} & B_{12} \\ B_{21} & O_{n+1,n+1} \end{pmatrix}, && C = \begin{pmatrix} O_{n,n} & O_{n,n+1} \\ \Phi & \Psi \end{pmatrix}, \\ &B_{12} = (I_n, O_{n1}), && B_{21} = \begin{pmatrix} -A_1 & O_{m,n-m} \\ O_{n-m,m} & -A_2 \\ O_{1m} & O_{1,n-m} \end{pmatrix}, \end{aligned} \tag{8}$$

O_{jk} is the $j\times k$ zero matrix, I_n is the $n\times n$ unit matrix, $\Phi=(\Phi_{jk})$, $\Psi=(\Psi_{jk})$, where the matrices Φ_{jk}, Ψ_{jk} are defined in (1). If we let $w(t)=Pz(-t)$, where the matrix P is defined by the transformation

$$\begin{aligned} &w_j(t) = z_j(-t), && w_{n+j}(t) = -z_{n+j}(-t), && j = 1, 2, \ldots, m, \\ &w_j(t) = -z_j(-t), && w_{n+j}(t) = z_{n+j}(-t), && j = m+1, \ldots, n, \\ &&& w_{2n+1}(t) = -z_{2n+1}(-t), \end{aligned}$$

then the differential equation (*) in Theorem 3 is given by

$$w' = A w + C^*(t, \lambda), \tag{9}$$

where

$$C^* = \begin{pmatrix} O_{n,n} & O_{n,n+1} \\ \Phi^* & \Psi^* \end{pmatrix}, \tag{10}$$

$$\Phi^*(t, \lambda) = [(-1)^{k+j}\Phi_{jk}(-t, \lambda), \quad j=1, 2, 3, \; k=1, 2],$$
$$\Psi^*(t, \lambda) = [(-1)^{k+j+1}\Psi_{jk}(-t, \lambda), \quad j, k = 1, 2, 3].$$

From the remark preceding Theorem 1, we may assume $\gamma=0$. If the assumptions of Theorem 1 are satisfied for $\gamma=0$, then, $\Phi^*(t,\lambda)=\Phi(t,\lambda)$, $\Psi^*(t,\lambda)=\Psi(t,\lambda)$. Therefore, Theorem 1 follows from Theorem 3 with $Q=I_{2n+1}$.

The most natural question to ask is whether it is possible to find matrices P, Q which are essentially different from the ones constructed for the proof of Theorem 1 and, therefore, obtain a boundedness theorem more general than Theorem 1. To investigate this question, it is clear that Q may be taken to be I_{2n+1} and it will be sufficient to find a non-singular matrix P which is independent of λ such that

$$P[A+C^*(t,\lambda)]\,P^{-1}=A+C(t,\lambda) \tag{11}$$

for all t and λ, where C^* is defined by (9) and (10) and C is defined by (8). It is convenient to let $P=(P_{jk})$, $j,k=1,2,3$ where each P_{jk} is a matrix and P_{11}, P_{22} are $n\times n$ and P_{33} is a scalar. Since (11) must be satisfied for all λ, this implies that $PA=AP$; or, $P_{11}=P_{22}$, $P_{21}=-P_{12}A_3$, $A_3=\operatorname{diag}(A_1,A_2)$, P_{11}, P_{12} diagonal, P_{33} arbitrary and all other P_{jk} are zero matrices. Furthermore, if each of the equations in (1) contains at least one nonzero periodic coefficient, then it is easy to show that P_{12} is the zero matrix. If we let $P_{11}=\operatorname{diag}(p_1,\ldots,p_n)$ $P_{33}=p_{n+1}$, $\Phi=(\varphi_{jk}, j=1,2,\ldots,n+1, k=1,2,\ldots,n)$, $\Psi=(\psi_{jk}, j,k=1,2,\ldots,n+1)$, then the relation $PC^*(t,\lambda)=C(t,\lambda)\,P$ leads to the system of equations

$$\begin{aligned} p_j\,\varphi_{jk}(-t,\lambda) &= \pm p_k\,\varphi_{jk}(t,\lambda), \qquad j=1,2,\ldots,n+1,\ k=1,2,\ldots,n,\\ p_j\,\psi_{jk}(-t,\lambda) &= \pm p_k\,\psi_{jk}(t,\lambda), \quad j,k=1,2,\ldots,n+1, \end{aligned} \tag{12}$$

for the undetermined quantities $p_1,\ldots,p_{n+1}$. If we let $-t=x$ in the first equation of (12), then we have

$$\varphi_{jk}(x,\lambda)=\pm\frac{p_k}{p_j}\,\varphi_{jk}(-x,\lambda)=\left(\frac{p_k}{p_j}\right)^2\varphi_{jk}(x,\lambda)$$

and, therefore, $(p_k/p_j)^2=1$, $j,k=1,2,\ldots,n+1$. By a change of the scale of the independent variable in (1), one can assume that each p_j, $j=1,2,\ldots,2n+1$, is either $+1$ or -1. But, with each p_j either $+1$ or -1, it is clear that there exists an elementary transformation which will make conditions (12) equivalent to the conditions in Theorem 1. Therefore, in this sense, Theorem 1 is the most general theorem for this class of functions which can be obtained by using Theorem 3.

The majority of this work was completed while the author was at Remington Rand Univac, St. Paul, Minnesota.

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Finite Integrity Bases for Five or Fewer Symmetric 3×3 Matrices

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1. Introduction

In a previous paper [*1*], finite integrity bases for five or fewer symmetric 3×3 matrices, under the orthogonal transformation group, have been derived. In the present paper, it will be shown that the integrity bases there derived are redundant in the sense that certain of their elements can be expressed as polynomials in the remaining elements, and integrity bases containing fewer elements will thus be obtained. The results of the previous paper [*1*], which form the starting point of the present paper, are essentially consequences of the Hamilton-Cayley theorem and the procedures adopted in the present paper are similarly based.

Finite integrity bases for five or fewer symmetric 3×3 matrices under the general linear transformation group have been given by TODD [*2*, *3*] and TURNBULL [*4*]. Since, by PEANO's theorem, a finite integrity basis for any number of symmetric matrices can be derived by polarization from an integrity basis for five symmetric 3×3 matrices together with the alternating invariant of six symmetric 3×3 matrices, TODD's result can be used, as he pointed out, to construct a finite integrity basis for any number of symmetric 3×3 matrices under the full linear group. In particular it can be used to construct a finite integrity basis for six symmetric 3×3 matrices under the full linear group. By replacing one of these matrices by the unit matrix, a finite integrity basis for five symmetric 3×3 matrices under the orthogonal group can be obtained in accordance with the Adjunction theorem (see, for example, WEYL [*5*]). The elements of the finite integrity basis thus obtained from TODD's results may, with considerable algebraic manipulation, be expressed as polynomials in traces of matrix products and a number of these traces which form an integrity basis may be selected by inspection. Certain of these may be expressed as polynomials in the remainder by employing the Hamilton-Cayley theorem, or results obtained therefrom by appropriate substitutions, and the end results of the present paper can be obtained. However, since the labour involved in obtaining these results from those of TODD is comparable with that of deriving them directly, we have adopted the latter course in the present paper.

2. Finite integrity bases for five or fewer matrices

In this section we give the integrity bases for one, two, three, four and five symmetric 3×3 matrices, under the orthogonal transformation group*, which

* It is immaterial whether we consider the full or the proper orthogonal group.

were obtained in the previous paper [*1*]. Some of the results, *e.g.* those for a single matrix and for two matrices are, of course, well-known.

Single matrix $\boldsymbol{a}$. $\quad \operatorname{tr}\boldsymbol{a}, \quad \operatorname{tr}\boldsymbol{a}^2, \quad \operatorname{tr}\boldsymbol{a}^3.$ (2.1)

Two matrices $\boldsymbol{a}, \boldsymbol{b}$.

$$\begin{array}{llll} \operatorname{tr}\boldsymbol{a}, & \operatorname{tr}\boldsymbol{a}^2, & \operatorname{tr}\boldsymbol{a}^3, & \\ \operatorname{tr}\boldsymbol{b}, & \operatorname{tr}\boldsymbol{b}^2, & \operatorname{tr}\boldsymbol{b}^3, & \\ \operatorname{tr}\boldsymbol{a}\boldsymbol{b}, & \operatorname{tr}\boldsymbol{a}\boldsymbol{b}^2, & \operatorname{tr}\boldsymbol{a}^2\boldsymbol{b}, & \operatorname{tr}\boldsymbol{a}^2\boldsymbol{b}^2. \end{array} \tag{2.2}$$

Three matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. The integrity bases for each of the three pairs of matrices which can be selected from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, together with the invariants

$$\operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}, \quad \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}^2 \quad \text{and} \quad \operatorname{tr}\boldsymbol{a}\boldsymbol{b}^2\boldsymbol{c}^2 \tag{2.3}$$

and the invariants which can be formed from (2.3) by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

Four matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$. The integrity bases for each of the four sets of three matrices which can be selected from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, together with the invariants

$$\operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}, \quad \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2, \quad \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}^2\boldsymbol{d}^2, \quad \operatorname{tr}\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}\boldsymbol{d}\boldsymbol{a}^2 \quad \text{and} \quad \operatorname{tr}\boldsymbol{b}\boldsymbol{a}\boldsymbol{c}\boldsymbol{d}\boldsymbol{a}^2 \tag{2.4}$$

and the invariants which can be obtained from these by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$.

Five matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$. The integrity bases for each of the five sets of four matrices which can be selected from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$, together with the invariants

$$\begin{gathered} \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}\boldsymbol{e}, \quad \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}\boldsymbol{e}^2, \quad \operatorname{tr}\boldsymbol{a}\boldsymbol{e}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}\boldsymbol{e}^2, \quad \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{e}\boldsymbol{c}\boldsymbol{d}\boldsymbol{e}^2, \\ \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{e}\boldsymbol{d}\boldsymbol{e}^2 \quad \text{and} \quad \operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2\boldsymbol{e}^2 \end{gathered} \tag{2.5}$$

and the invariants obtained from these by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$.

3. Some theorems concerning traces of matrix products

In § 4 and the following sections, we shall show that certain of the elements of the integrity bases for three, four and five matrices given in § 2 can be expressed as polynomials in the remaining elements. In this way we shall derive integrity bases for three, four and five matrices containing fewer elements than those given in § 2. In order to do this we shall require certain results which are either stated or derived in a previous paper [*1*]. These are given in the present section.

Firstly, we require the following two lemmas which result immediately from the definition of matrix multiplication.

Lemma 1. *The trace of a matrix product formed from* 3×3 *matrices is unaltered by cyclic permutation of the factors in the product.*

Lemma 2. *The trace of a matrix product formed from symmetric* 3×3 *matrices is unaltered if the order of the factors in the product is reversed.*

If I_1 and I_2 are two scalar invariants of any number of 3×3 matrices of equal partial degrees in each of the matrices and $I_1 - I_2$ is expressible as a polynomial in invariants of lower or equal partial degrees in each of the matrices and of lower total degree, we say I_1 and I_2 are *equivalent* and write

$$I_1 \doteq I_2.$$

We also say $I_1 - I_2$ is *reducible* and write

$$I_1 - I_2 \equiv 0.$$

We require the following relations which, in the previous paper [*1*], are seen to result from the Hamilton-Cayley theorem. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be three 3×3 matrices and let $\boldsymbol{y}$ and $\boldsymbol{z}$ be any 3×3 matrices (in particular cases $\boldsymbol{I}$). Then, we have

$$\begin{aligned}\operatorname{tr}\boldsymbol{y\,a\,b\,c\,z} + \operatorname{tr}\boldsymbol{y\,b\,c\,a\,z} + \operatorname{tr}\boldsymbol{y\,c\,a\,b\,z} &+ \\ + \operatorname{tr}\boldsymbol{y\,b\,a\,c\,z} + \operatorname{tr}\boldsymbol{y\,a\,c\,b\,z} &+ \operatorname{tr}\boldsymbol{y\,c\,b\,a\,z} \equiv 0,\end{aligned} \tag{3.1}$$

unless $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{I}$, and

$$\operatorname{tr}\boldsymbol{y\,a\,b\,a^2 z} + \operatorname{tr}\boldsymbol{y\,a^2 b\,a\,z} \equiv 0, \tag{3.2}$$

even if $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{I}$.

Also, if $\boldsymbol{x}$ $(\neq \boldsymbol{I})$ is a 3×3 matrix, we have

$$\operatorname{tr}\boldsymbol{y\,a^2 x\,b^2 z} \equiv 0, \tag{3.3}$$

unless $\boldsymbol{y} = \boldsymbol{z} = \boldsymbol{I}$.

4. Reduction of the integrity basis for three matrices

We take as our starting point the finite integrity basis for the three symmetric 3×3 matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, given in § 2, and consider equivalences of the invariants (2.3) and the invariants formed from them by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

From lemmas 1 and 2, it is immediately obvious that $\operatorname{tr}\boldsymbol{abc}$ is unaltered by permutation of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

In a similar manner it follows from Lemmas 1 and 2 that

$$\operatorname{tr}\boldsymbol{a\,b^2 c^2} = \operatorname{tr}\boldsymbol{a\,c^2 b^2}$$

and

$$\operatorname{tr}\boldsymbol{a\,b\,c^2} = \operatorname{tr}\boldsymbol{b\,a\,c^2}.$$

We thus see that an integrity basis for the three symmetric 3×3 matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ under the full or proper orthogonal group is formed by the integrity bases, given by (2.2), for the three sets of two matrices which can be selected from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, together with the invariants

$$\begin{gathered}\operatorname{tr}\boldsymbol{a\,b\,c},\quad \operatorname{tr}\boldsymbol{a\,b\,c^2},\quad \operatorname{tr}\boldsymbol{b\,c\,a^2},\quad \operatorname{tr}\boldsymbol{c\,a\,b^2},\quad \operatorname{tr}\boldsymbol{a\,b^2 c^2},\\ \operatorname{tr}\boldsymbol{b\,c^2 a^2}\quad \text{and}\quad \operatorname{tr}\boldsymbol{c\,a^2 b^2}.\end{gathered} \tag{4.1}$$

5. Reduction of the integrity basis for four matrices

The elements of a finite integrity basis for four symmetric 3×3 matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, which involve all four of the matrices, are given by (2.4) and the invariants formed from these by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$. We shall consider in turn each invariant listed in (2.4) together with the invariants formed from it by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, with a view to determining which of these may be omitted from the integrity basis in the light of the results given in § 3.

$\operatorname{tr}\boldsymbol{abcd}$. From Lemmas 1 and 2 we see that each of the invariants formed by permutation of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in $\operatorname{tr}\boldsymbol{abcd}$ is equal to either $\operatorname{tr}\boldsymbol{abcd}$, $\operatorname{tr}\boldsymbol{abdc}$ or

tr $\boldsymbol{acbd}$. From (3.1), replacing $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ by $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ respectively and taking $\boldsymbol{y}=\boldsymbol{a}$, $\boldsymbol{z}=\boldsymbol{I}$, we obtain

$$2(\operatorname{tr} \boldsymbol{a\,b\,c\,d} + \operatorname{tr} \boldsymbol{a\,b\,d\,c} + \operatorname{tr} \boldsymbol{a\,c\,b\,d}) \equiv 0. \tag{5.1}$$

Thus, we can omit one of the three invariants on the left-hand side (say tr $\boldsymbol{acbd}$) from the integrity basis and, of the terms tr $\boldsymbol{abcd}$ and the terms formed from them by permutation of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, include only

$$\operatorname{tr} \boldsymbol{a\,b\,c\,d} \quad \text{and} \quad \operatorname{tr} \boldsymbol{a\,b\,d\,c}. \tag{5.2}$$

tr $\boldsymbol{abcd}^2$. From lemmas 1 and 2, we see that each of the invariants formed by permutation of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in tr $\boldsymbol{abcd}^2$ is equal to either tr $\boldsymbol{abcd}^2$, tr $\boldsymbol{acbd}^2$ or tr $\boldsymbol{bacd}^2$. From (3.1), taking $\boldsymbol{y}=\boldsymbol{I}$, $\boldsymbol{z}=\boldsymbol{d}^2$, we have

$$2(\operatorname{tr} \boldsymbol{a\,b\,c\,d}^2 + \operatorname{tr} \boldsymbol{a\,c\,b\,d}^2 + \operatorname{tr} \boldsymbol{b\,a\,c\,d}^2) \equiv 0. \tag{5.3}$$

Thus, we may omit one of the invariants on the left-hand side of (5.3), say tr $\boldsymbol{bacd}^2$, from the integrity basis and, of the terms tr $\boldsymbol{abcd}^2$ and invariants formed from it by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, include only

$$\operatorname{tr} \boldsymbol{a\,b\,c\,d}^2 \quad \text{and} \quad \operatorname{tr} \boldsymbol{a\,c\,b\,d}^2. \tag{5.4}$$

Applying similar arguments to the invariants formed by permutations of $\boldsymbol{d}, \boldsymbol{a}, \boldsymbol{b}$ in tr $\boldsymbol{dabc}^2$, of $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{a}$ in tr $\boldsymbol{cdab}^2$ and of $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in tr $\boldsymbol{bcda}^2$, we see that of the invariants obtained from tr $\boldsymbol{abcd}^2$ by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, we may omit from the integrity basis all except

$$\begin{matrix} \operatorname{tr} \boldsymbol{a\,b\,c\,d}^2, & \operatorname{tr} \boldsymbol{a\,c\,b\,d}^2, & \operatorname{tr} \boldsymbol{d\,a\,b\,c}^2, & \operatorname{tr} \boldsymbol{d\,b\,a\,c}^2, \\ \operatorname{tr} \boldsymbol{c\,d\,a\,b}^2, & \operatorname{tr} \boldsymbol{c\,a\,d\,b}^2, & \operatorname{tr} \boldsymbol{b\,c\,d\,a}^2, & \operatorname{tr} \boldsymbol{b\,d\,c\,a}^2. \end{matrix} \tag{5.5}$$

tr $\boldsymbol{abc}^2\boldsymbol{d}^2$. Now, we consider the invariant tr $\boldsymbol{abc}^2\boldsymbol{d}^2$ and invariants formed from it by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$.

Replacing $\boldsymbol{c}$ and $\boldsymbol{d}$ by $\boldsymbol{c}^2$ and $\boldsymbol{d}^2$ in (5.1), we obtain

$$\operatorname{tr} \boldsymbol{a\,b\,c}^2\boldsymbol{d}^2 + \operatorname{tr} \boldsymbol{a\,b\,d}^2\boldsymbol{c}^2 + \operatorname{tr} \boldsymbol{a\,c}^2\boldsymbol{b\,d}^2 \equiv 0. \tag{5.6}$$

Now, taking $\boldsymbol{y}=\boldsymbol{a}$, $\boldsymbol{z}=\boldsymbol{I}$ in (3.3) and replacing $\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{b}$ by $\boldsymbol{c}, \boldsymbol{b}, \boldsymbol{d}$ respectively we have

$$\operatorname{tr} \boldsymbol{a\,c}^2\boldsymbol{b\,d}^2 \equiv 0. \tag{5.7}$$

From (5.6) and (5.7), we obtain

$$\operatorname{tr} \boldsymbol{a\,b\,c}^2\boldsymbol{d}^2 \equiv -\operatorname{tr} \boldsymbol{a\,b\,d}^2\boldsymbol{c}^2. \tag{5.8}$$

Employing Lemmas 1 and 2, we obtain

$$\operatorname{tr} \boldsymbol{c\,b\,d}^2\boldsymbol{c}^2 = \operatorname{tr} \boldsymbol{b\,a\,c}^2\boldsymbol{d}^2. \tag{5.9}$$

From relations of the types (5.8) and (5.9), we see that of the invariants formed by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in tr $\boldsymbol{abc}^2\boldsymbol{d}^2$, we may omit from the integrity basis all except

$$\begin{matrix} \operatorname{tr} \boldsymbol{a\,b\,c}^2\boldsymbol{d}^2, & \operatorname{tr} \boldsymbol{a\,c\,b}^2\boldsymbol{d}^2, & \operatorname{tr} \boldsymbol{a\,d\,b}^2\boldsymbol{c}^2, \\ \operatorname{tr} \boldsymbol{b\,c\,a}^2\boldsymbol{d}^2, & \operatorname{tr} \boldsymbol{b\,d\,a}^2\boldsymbol{c}^2, & \operatorname{tr} \boldsymbol{c\,d\,a}^2\boldsymbol{b}^2. \end{matrix} \tag{5.10}$$

$\operatorname{tr} \boldsymbol{bacda}^2$ and $\operatorname{tr} \boldsymbol{bcada}^2$. Finally, we consider the invariants $\operatorname{tr} \boldsymbol{bacda}^2$ and $\operatorname{tr} \boldsymbol{bcada}^2$ and the invariants obtained from them by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$.

Taking $\boldsymbol{y} = \boldsymbol{I}$ and $\boldsymbol{z} = \boldsymbol{da}^2$ in (3.1), we obtain

$$\operatorname{tr}(\boldsymbol{abc} + \boldsymbol{acb} + \boldsymbol{bac} + \boldsymbol{bca} + \boldsymbol{cab} + \boldsymbol{cba})\,\boldsymbol{da}^2 \equiv 0.$$

Since $\operatorname{tr} \boldsymbol{abcda}^2 \equiv 0$ and $\operatorname{tr} \boldsymbol{acbda}^2 \equiv 0$, we have

$$\operatorname{tr}(\boldsymbol{bacda}^2 + \boldsymbol{bcada}^2 + \boldsymbol{cabda}^2 + \boldsymbol{cbada}^2) \equiv 0. \tag{5.11}$$

Now, from Lemmas 1 and 2, we have

$$\operatorname{tr} \boldsymbol{bcada}^2 = \operatorname{tr} \boldsymbol{ada}^2\boldsymbol{bc} = \operatorname{tr} \boldsymbol{cba}^2\boldsymbol{da}. \tag{5.12}$$

From (3.2), replacing $\boldsymbol{b}$ by $\boldsymbol{d}$ and taking $\boldsymbol{y} = \boldsymbol{cb}$, $\boldsymbol{z} = \boldsymbol{I}$, we obtain

$$\operatorname{tr} \boldsymbol{cba}^2\boldsymbol{da} + \operatorname{tr} \boldsymbol{cbada}^2 \equiv 0. \tag{5.13}$$

From (5.12) and (5.13), we obtain

$$\operatorname{tr} \boldsymbol{bcada}^2 \equiv -\operatorname{tr} \boldsymbol{cbada}^2. \tag{5.14}$$

Employing the relation (5.14) in (5.11), we obtain

$$\operatorname{tr} \boldsymbol{bacda}^2 \equiv -\operatorname{tr} \boldsymbol{cabda}^2. \tag{5.15}$$

Again employing Lemma 1 and the relation (3.2), we obtain

$$\operatorname{tr} \boldsymbol{bacda}^2 \equiv \operatorname{tr} \boldsymbol{cda}^2\boldsymbol{ba} \equiv -\operatorname{tr} \boldsymbol{cdaba}^2. \tag{5.16}$$

From (5.16), we see that we may omit from the integrity basis $\operatorname{tr} \boldsymbol{bcada}^2$ and invariants formed from it by permuting $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$.

From (5.14) and (5.16), we have

$$\operatorname{tr} \boldsymbol{bacda}^2 \equiv -\operatorname{tr} \boldsymbol{badca}^2. \tag{5.17}$$

From (5.15) and (5.17) we see that we may omit from the integrity basis all but one of the set of invariants consisting of $\operatorname{tr} \boldsymbol{bacda}^2$ and the invariants formed by permuting $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in this.

Thus, of the invariants $\operatorname{tr} \boldsymbol{bacda}^2$ and $\operatorname{tr} \boldsymbol{bcada}^2$ and the invariants obtained from these by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, we may omit from the integrity basis all except

$$\operatorname{tr} \boldsymbol{bacda}^2, \quad \operatorname{tr} \boldsymbol{cbdab}^2, \quad \operatorname{tr} \boldsymbol{dcabc}^2 \quad \text{and} \quad \operatorname{tr} \boldsymbol{adbcd}^2. \tag{5.18}$$

Collecting the results obtained above, we see that an integrity basis for the four 3×3 symmetric matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ under the full or proper orthogonal groups is formed by the integrity bases for the four sets of three matrices which can be selected from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, together with

(i) $\operatorname{tr} \boldsymbol{abcd}$, $\quad \operatorname{tr} \boldsymbol{abdc}$;

(ii) $\operatorname{tr} \boldsymbol{abcd}^2$, $\quad \operatorname{tr} \boldsymbol{acbd}^2$, $\quad \operatorname{tr} \boldsymbol{dabc}^2$, $\quad \operatorname{tr} \boldsymbol{dbac}^2$,
$\operatorname{tr} \boldsymbol{cdab}^2$, $\quad \operatorname{tr} \boldsymbol{cadb}^2$, $\quad \operatorname{tr} \boldsymbol{bcda}^2$, $\quad \operatorname{tr} \boldsymbol{bdca}^2$;

(iii) $\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}^2\boldsymbol{d}^2$, $\operatorname{tr}\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{b}^2\boldsymbol{d}^2$, $\operatorname{tr}\boldsymbol{a}\,\boldsymbol{d}\,\boldsymbol{b}^2\boldsymbol{c}^2$,
$\operatorname{tr}\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{a}^2\boldsymbol{d}^2$, $\operatorname{tr}\boldsymbol{b}\,\boldsymbol{d}\,\boldsymbol{a}^2\boldsymbol{c}^2$, $\operatorname{tr}\boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{a}^2\boldsymbol{b}^2$;

(iv) $\operatorname{tr}\boldsymbol{b}\,\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{a}^2$, $\operatorname{tr}\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{d}\,\boldsymbol{a}\,\boldsymbol{b}^2$, $\operatorname{tr}\boldsymbol{d}\,\boldsymbol{c}\,\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}^2$ and $\operatorname{tr}\boldsymbol{a}\,\boldsymbol{d}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d}^2$.

6. Reduction of the integrity basis for five matrices

In this section we shall discuss the equivalence of the elements, given in § 2, of the finite integrity basis for the five symmetric 3×3 matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, which involve all five of the matrices. The procedure will be similar to that adopted in § 4 in discussing the case of four matrices.

$\operatorname{tr}\boldsymbol{abcde}$. We consider first the invariants formed from $\operatorname{tr}\boldsymbol{abcde}$ by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$. Taking $\boldsymbol{y}=\boldsymbol{de}$ and $\boldsymbol{z}=\boldsymbol{I}$ in (3.1), and employing the notation

$$\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}=\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}+\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{b}+\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{a}+\boldsymbol{b}\,\boldsymbol{a}\,\boldsymbol{c}+\boldsymbol{c}\,\boldsymbol{a}\,\boldsymbol{b}+\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{a}, \tag{6.1}$$

we obtain

$$\operatorname{tr}(\boldsymbol{d}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c})\equiv 0. \tag{6.2}$$

Noting that $\sum\boldsymbol{abc}$ is a symmetric matrix, we have, with Lemmas 1 and 2,

$$\begin{aligned}\operatorname{tr}(\boldsymbol{d}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}) = \operatorname{tr}(\boldsymbol{e}\,\boldsymbol{d}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}) &= \operatorname{tr}\{\boldsymbol{e}(\textstyle\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c})\,\boldsymbol{d}\}\\ &= \operatorname{tr}\{\boldsymbol{d}(\textstyle\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c})\,\boldsymbol{e}\}\equiv 0.\end{aligned} \tag{6.3}$$

We can form ten relations of the type (6.2). These are

$$\begin{array}{lll}\operatorname{tr}(\boldsymbol{d}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c})\equiv 0, & \operatorname{tr}(\boldsymbol{c}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{d})\equiv 0, & \operatorname{tr}(\boldsymbol{c}\,\boldsymbol{d}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{e})\equiv 0,\\ \operatorname{tr}(\boldsymbol{b}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{d})\equiv 0, & \operatorname{tr}(\boldsymbol{b}\,\boldsymbol{d}\sum \boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{e})\equiv 0, & \operatorname{tr}(\boldsymbol{b}\,\boldsymbol{c}\sum \boldsymbol{a}\,\boldsymbol{d}\,\boldsymbol{e})\equiv 0,\\ \operatorname{tr}(\boldsymbol{a}\,\boldsymbol{e}\sum \boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d})\equiv 0, & \operatorname{tr}(\boldsymbol{a}\,\boldsymbol{d}\sum \boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{e})\equiv 0, & \operatorname{tr}(\boldsymbol{a}\,\boldsymbol{c}\sum \boldsymbol{b}\,\boldsymbol{d}\,\boldsymbol{e})\equiv 0,\\ \operatorname{tr}(\boldsymbol{a}\,\boldsymbol{b}\sum \boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e})\equiv 0. & & \end{array} \tag{6.4}$$

Not all of the relations (6.3) are independent, for it may be readily verified by introducing relations of the type (6.1) that

$$\begin{aligned}&\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{c}\sum \boldsymbol{b}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{d}\sum \boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{e}\sum \boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d})\\ &\qquad=\operatorname{tr}(\boldsymbol{b}\,\boldsymbol{c}\sum \boldsymbol{a}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{b}\,\boldsymbol{d}\sum \boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{b}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{d}),\\ &\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{b}\sum \boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{d}\sum \boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{e}\sum \boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d})\\ &\qquad=\operatorname{tr}(\boldsymbol{c}\,\boldsymbol{b}\sum \boldsymbol{a}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{c}\,\boldsymbol{d}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{c}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{d}),\\ &\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{b}\sum \boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{c}\sum \boldsymbol{b}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{e}\sum \boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d})\\ &\qquad=\operatorname{tr}(\boldsymbol{d}\,\boldsymbol{b}\sum \boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{d}\,\boldsymbol{c}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{d}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}),\\ &\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{b}\sum \boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{c}\sum \boldsymbol{b}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{a}\,\boldsymbol{d}\sum \boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{e})\\ &\qquad=\operatorname{tr}(\boldsymbol{e}\,\boldsymbol{b}\sum \boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{d})+\operatorname{tr}(\boldsymbol{e}\,\boldsymbol{c}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{d})+\operatorname{tr}(\boldsymbol{e}\,\boldsymbol{d}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}).\end{aligned} \tag{6.5}$$

From equations (6.5), we can derive a number of similar equations. For example, subtracting the second equation from the first, we obtain

$$\begin{aligned}&\operatorname{tr}(\boldsymbol{b}\,\boldsymbol{a}\sum \boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{b}\,\boldsymbol{d}\sum \boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{b}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{d})\\ &\qquad=\operatorname{tr}(\boldsymbol{c}\,\boldsymbol{a}\sum \boldsymbol{b}\,\boldsymbol{d}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{c}\,\boldsymbol{d}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{e})+\operatorname{tr}(\boldsymbol{c}\,\boldsymbol{e}\sum \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{d}).\end{aligned} \tag{6.6}$$

The four equations (6.5), regarded as equations in which the unknowns are $\mathrm{tr}(\boldsymbol{ab}\sum\boldsymbol{cde})$, $\mathrm{tr}(\boldsymbol{ac}\sum\boldsymbol{bde})$, $\mathrm{tr}(\boldsymbol{ad}\sum\boldsymbol{bce})$ and $\mathrm{tr}(\boldsymbol{ae}\sum\boldsymbol{bcd})$, are independent and may therefore be used to determine these quantities in terms of $\mathrm{tr}(\boldsymbol{bc}\sum\boldsymbol{ade})$, $\mathrm{tr}(\boldsymbol{bd}\sum\boldsymbol{ace})$, $\mathrm{tr}(\boldsymbol{be}\sum\boldsymbol{acd})$, $\mathrm{tr}(\boldsymbol{cd}\sum\boldsymbol{abe})$, $\mathrm{tr}(\boldsymbol{ce}\sum\boldsymbol{abd})$ and $\mathrm{tr}(\boldsymbol{de}\sum\boldsymbol{abc})$. Thus, the last four of the relations (6.4) may be regarded as consequences of the first six.

These first six relations may be re-written, using Lemmas 1 and 2, as

$$\left.\begin{aligned}
&\mathrm{tr}\,\boldsymbol{abcde}+\mathrm{tr}\,\boldsymbol{abced}+\mathrm{tr}\,\boldsymbol{abdec}+\mathrm{tr}\,\boldsymbol{abedc}+\mathrm{tr}\,\boldsymbol{acbde}+\mathrm{tr}\,\boldsymbol{acbed}=0,\\
&\mathrm{tr}\,\boldsymbol{abced}+\mathrm{tr}\,\boldsymbol{abdce}+\mathrm{tr}\,\boldsymbol{abdec}+\mathrm{tr}\,\boldsymbol{abecd}+\mathrm{tr}\,\boldsymbol{acebd}+\mathrm{tr}\,\boldsymbol{adbce}=0,\\
&\mathrm{tr}\,\boldsymbol{abcde}+\mathrm{tr}\,\boldsymbol{abdce}+\mathrm{tr}\,\boldsymbol{abecd}+\mathrm{tr}\,\boldsymbol{abedc}+\mathrm{tr}\,\boldsymbol{acdbe}+\mathrm{tr}\,\boldsymbol{adcbe}=0,\\
&\mathrm{tr}\,\boldsymbol{abecd}+\mathrm{tr}\,\boldsymbol{abedc}+\mathrm{tr}\,\boldsymbol{acbed}+\mathrm{tr}\,\boldsymbol{acdbe}+\mathrm{tr}\,\boldsymbol{acebd}+\mathrm{tr}\,\boldsymbol{adcbe}=0,\\
&\mathrm{tr}\,\boldsymbol{abdce}+\mathrm{tr}\,\boldsymbol{abdec}+\mathrm{tr}\,\boldsymbol{acbde}+\mathrm{tr}\,\boldsymbol{acdbe}+\mathrm{tr}\,\boldsymbol{acebd}+\mathrm{tr}\,\boldsymbol{adbce}=0,\\
&\mathrm{tr}\,\boldsymbol{abcde}+\mathrm{tr}\,\boldsymbol{abced}+\mathrm{tr}\,\boldsymbol{acbde}+\mathrm{tr}\,\boldsymbol{acbed}+\mathrm{tr}\,\boldsymbol{adbce}+\mathrm{tr}\,\boldsymbol{adcbe}=0.
\end{aligned}\right\}\tag{6.7}$$

Regarding these equations as a set of six simultaneous equations in which the unknowns are $\mathrm{tr}\,\boldsymbol{abced}$, $\mathrm{tr}\,\boldsymbol{abdce}$, $\mathrm{tr}\,\boldsymbol{abedc}$, $\mathrm{tr}\,\boldsymbol{acbde}$, $\mathrm{tr}\,\boldsymbol{acebd}$ and $\mathrm{tr}\,\boldsymbol{adcbe}$, we can show, by considering the rank of the matrix of the coefficients of the unknowns, that they can be used to determine the six invariants in terms of $\mathrm{tr}\,\boldsymbol{abcde}$, $\mathrm{tr}\,\boldsymbol{abdec}$, $\mathrm{tr}\,\boldsymbol{abecd}$, $\mathrm{tr}\,\boldsymbol{acdbe}$, $\mathrm{tr}\,\boldsymbol{acbed}$ and $\mathrm{tr}\,\boldsymbol{adbce}$ and invariants involving four or fewer of the five matrices. Thus, of the invariants formed by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ in $\mathrm{tr}\,\boldsymbol{abcde}$, we may omit from the integrity basis all except

$$\begin{gathered}
\mathrm{tr}\,\boldsymbol{abcde},\quad \mathrm{tr}\,\boldsymbol{abdec},\quad \mathrm{tr}\,\boldsymbol{abecd},\\
\mathrm{tr}\,\boldsymbol{acdbe},\quad \mathrm{tr}\,\boldsymbol{acbed},\quad \mathrm{tr}\,\boldsymbol{adbce}.
\end{gathered}\tag{6.8}$$

The set of invariants (6.8) are all formed from $\mathrm{tr}\,\boldsymbol{abcde}$ by even permutations of the matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ and it is easily shown that any invariant formed from $\mathrm{tr}\,\boldsymbol{abcde}$ by even permutation of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ may, by means of Lemmas 1 and 2, be expressed in one or other of the forms (6.8). The choice of invariants to be retained in the integrity basis is by no means unique; for example, in place of the set (6.8), we might have chosen to retain the invariants formed by odd permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ in $\mathrm{tr}\,\boldsymbol{abcde}$, and other choices are also possible.

$\mathrm{tr}\,\boldsymbol{abcde}^2$. We next consider the invariants formed from $\mathrm{tr}\,\boldsymbol{abcde}^2$ by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$.

Replacing $\boldsymbol{e}$ by $\boldsymbol{e}^2$ in the above discussion, we see that each of the invariants formed from $\mathrm{tr}\,\boldsymbol{abcde}^2$ by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ can be expressed as a polynomial in

$$\begin{gathered}
\mathrm{tr}\,\boldsymbol{abcde}^2,\quad \mathrm{tr}\,\boldsymbol{abde}^2\boldsymbol{c},\quad \mathrm{tr}\,\boldsymbol{abe}^2\boldsymbol{cd},\\
\mathrm{tr}\,\boldsymbol{acdbe}^2,\quad \mathrm{tr}\,\boldsymbol{acbe}^2\boldsymbol{d},\quad \mathrm{tr}\,\boldsymbol{adbce}^2.
\end{gathered}\tag{6.9}$$

Thus, noting that $\mathrm{tr}\,\boldsymbol{abde}^2\boldsymbol{c}=\mathrm{tr}\,\boldsymbol{cabde}^2$, $\mathrm{tr}\,\boldsymbol{abe}^2\boldsymbol{cd}=\mathrm{tr}\,\boldsymbol{cdabe}^2$ and $\mathrm{tr}\,\boldsymbol{acbe}^2\boldsymbol{d}=\mathrm{tr}\,\boldsymbol{dacbe}^2$, we see that of the invariants formed from $\mathrm{tr}\,\boldsymbol{abcde}^2$ by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$, we may omit from the integrity basis all except

$$\begin{gathered}
\mathrm{tr}\,\boldsymbol{abcde}^2,\quad \mathrm{tr}\,\boldsymbol{cabde}^2,\quad \mathrm{tr}\,\boldsymbol{cdabe}^2,\\
\mathrm{tr}\,\boldsymbol{acdbe}^2,\quad \mathrm{tr}\,\boldsymbol{dacbe}^2,\quad \mathrm{tr}\,\boldsymbol{adbce}^2,
\end{gathered}\tag{6.10}$$

and invariants formed from these by permuting $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ cyclically.

$\operatorname{tr} \boldsymbol{abcede}^2$, $\operatorname{tr} \boldsymbol{abecde}^2$, $\operatorname{tr} \boldsymbol{aebcde}^2$. Now, consider the invariants formed by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ in $\operatorname{tr} \boldsymbol{abcede}^2$, $\operatorname{tr} \boldsymbol{abecde}^2$ and $\operatorname{tr} \boldsymbol{aebcde}^2$.

Replacing $\boldsymbol{a}$ and $\boldsymbol{b}$ in (3.2) by $\boldsymbol{e}$ and $\boldsymbol{acd}$ respectively and taking $\boldsymbol{y}=\boldsymbol{b}$, $\boldsymbol{z}=\boldsymbol{I}$, we obtain, with Lemma 1,

$$\operatorname{tr} \boldsymbol{b\,e\,a\,c\,d\,e}^2 \equiv -\operatorname{tr} \boldsymbol{b\,e}^2\boldsymbol{a\,c\,d\,e} = -\operatorname{tr} \boldsymbol{a\,c\,d\,e\,b\,e}^2. \tag{6.11}$$

It follows from (6.11) and relations obtained from it by permuting $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ that, of the invariants formed by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in $\operatorname{tr} \boldsymbol{abcede}^2$, $\operatorname{tr} \boldsymbol{abecde}^2$ and $\operatorname{tr} \boldsymbol{aebcde}^2$, we may omit from the integrity basis all except those formed by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in $\operatorname{tr} \boldsymbol{abecde}^2$ and $\operatorname{tr} \boldsymbol{abcede}^2$.

Replacing $\boldsymbol{a}$ and $\boldsymbol{b}$ in (3.2) by $\boldsymbol{e}$ and $\boldsymbol{cd}$ respectively and taking $\boldsymbol{y}=\boldsymbol{ba}$, $\boldsymbol{z}=\boldsymbol{I}$, we obtain, with Lemma 1,

$$\operatorname{tr} \boldsymbol{b\,a\,e\,c\,d\,e}^2 \equiv -\operatorname{tr} \boldsymbol{b\,a\,e}^2\boldsymbol{c\,d\,e} = -\operatorname{tr} \boldsymbol{c\,d\,e\,b\,a\,e}^2. \tag{6.12}$$

Using Lemmas 1 and 2, we can obtain

$$\operatorname{tr} \boldsymbol{c\,d\,e\,b\,a\,e}^2 = \operatorname{tr} \boldsymbol{e}^2\boldsymbol{c\,d\,e\,b\,a} = \operatorname{tr} \boldsymbol{a\,b\,e\,d\,c\,e}^2. \tag{6.13}$$

From (6.12) and (6.13) and relations obtained from these by permuting $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, we see that, of the twenty-four invariants obtained by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in $\operatorname{tr} \boldsymbol{abecde}^2$, we may omit from the integrity basis all except six, which may be chosen as

$$\begin{gathered}\operatorname{tr} \boldsymbol{a\,b\,e\,c\,d\,e}^2, \quad \operatorname{tr} \boldsymbol{b\,a\,e\,c\,d\,e}^2, \quad \operatorname{tr} \boldsymbol{a\,c\,e\,b\,d\,e}^2, \quad \operatorname{tr} \boldsymbol{c\,a\,e\,b\,d\,e}^2, \\ \operatorname{tr} \boldsymbol{b\,c\,e\,a\,d\,e}^2 \quad \text{and} \quad \operatorname{tr} \boldsymbol{c\,b\,e\,a\,d\,e}^2.\end{gathered} \tag{6.14}$$

Replacing $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in (3.1) by $\boldsymbol{a}, \boldsymbol{e}, \boldsymbol{d}$ respectively and taking $\boldsymbol{y}=\boldsymbol{I}$ and $\boldsymbol{z}=\boldsymbol{bce}^2$, we obtain

$$\operatorname{tr}(\boldsymbol{a\,e\,d}+\boldsymbol{a\,d\,e}+\boldsymbol{d\,a\,e}+\boldsymbol{d\,e\,a}+\boldsymbol{e\,a\,d}+\boldsymbol{e\,d\,a})\,\boldsymbol{b\,c\,e}^2 \equiv 0. \tag{6.15}$$

Since $\operatorname{tr} \boldsymbol{eadbce}^2 \equiv 0$ and $\operatorname{tr} \boldsymbol{edabce}^2 \equiv 0$, the relation (6.15) yields

$$\operatorname{tr} \boldsymbol{a\,e\,d\,b\,c\,e}^2+\operatorname{tr} \boldsymbol{a\,d\,e\,b\,c\,e}^2+\operatorname{tr} \boldsymbol{d\,a\,e\,b\,c\,e}^2+\operatorname{tr} \boldsymbol{d\,e\,a\,b\,c\,e}^2 \equiv 0. \tag{6.16}$$

By permutation of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in (6.11), we obtain

$$\operatorname{tr} \boldsymbol{a\,e\,d\,b\,c\,e}^2 \equiv -\operatorname{tr} \boldsymbol{d\,b\,c\,e\,a\,e}^2 \quad \text{and} \quad \operatorname{tr} \boldsymbol{d\,e\,a\,b\,c\,e}^2 \equiv -\operatorname{tr} \boldsymbol{a\,b\,c\,e\,d\,e}^2. \tag{6.17}$$

From (6.16) and (6.17), we have

$$\operatorname{tr} \boldsymbol{a\,d\,e\,b\,c\,e}^2+\operatorname{tr} \boldsymbol{d\,a\,e\,b\,c\,e}^2-\operatorname{tr} \boldsymbol{d\,b\,c\,e\,a\,e}^2-\operatorname{tr} \boldsymbol{a\,b\,c\,e\,d\,e}^2 \equiv 0. \tag{6.18}$$

From (6.12) and (6.13), we have

$$\operatorname{tr} \boldsymbol{a\,b\,e\,d\,c\,e}^2 \equiv -\operatorname{tr} \boldsymbol{b\,a\,e\,c\,d\,e}^2. \tag{6.19}$$

By permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in (6.19), we obtain

$$\operatorname{tr} \boldsymbol{a\,d\,e\,b\,c\,e}^2 \equiv -\operatorname{tr} \boldsymbol{d\,a\,e\,c\,b\,e}^2 \quad \text{and} \quad \operatorname{tr} \boldsymbol{d\,a\,e\,b\,c\,e}^2 \equiv -\operatorname{tr} \boldsymbol{a\,d\,e\,c\,b\,e}^2. \tag{6.20}$$

Permuting $\boldsymbol{b}$ and $\boldsymbol{c}$ in (6.18) and employing the relations (6.20), we obtain

$$-\operatorname{tr} \boldsymbol{d\,a\,e\,b\,c\,e}^2-\operatorname{tr} \boldsymbol{a\,d\,e\,b\,c\,e}^2-\operatorname{tr} \boldsymbol{d\,c\,b\,e\,a\,e}^2-\operatorname{tr} \boldsymbol{a\,c\,b\,e\,d\,e}^2 \equiv 0. \tag{6.21}$$

The relations (6.18) and (6.21) yield

$$\operatorname{tr} \boldsymbol{d\,b\,c\,e\,a\,e}^2+\operatorname{tr} \boldsymbol{a\,b\,c\,e\,d\,e}^2+\operatorname{tr} \boldsymbol{d\,c\,b\,e\,a\,e}^2+\operatorname{tr} \boldsymbol{a\,c\,b\,e\,d\,e}^2 \equiv 0. \tag{6.22}$$

From Lemmas 1 and 2 we have

$$\operatorname{tr} \boldsymbol{a\,b\,c\,e\,d\,e}^2 = \operatorname{tr} \boldsymbol{d\,e\,c\,b\,a\,e}^2 \quad \text{and} \quad \operatorname{tr} \boldsymbol{a\,c\,b\,e\,d\,e}^2 = \operatorname{tr} \boldsymbol{d\,e\,b\,c\,a\,e}^2. \tag{6.23}$$

With (6.23), we may re-write (6.22) as

$$\operatorname{tr}\left\{\boldsymbol{d}(\boldsymbol{b\,c\,e} + \boldsymbol{e\,c\,b} + \boldsymbol{c\,b\,e} + \boldsymbol{e\,b\,c})\,\boldsymbol{a\,e}^2\right\} \equiv 0. \tag{6.24}$$

Now replacing $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in (3.1) by $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}$ respectively and taking $\boldsymbol{y} = \boldsymbol{d}$ and $\boldsymbol{z} = \boldsymbol{a\,e}^2$, we obtain

$$\operatorname{tr}\left\{\boldsymbol{d}(\boldsymbol{b\,c\,e} + \boldsymbol{e\,c\,b} + \boldsymbol{c\,b\,e} + \boldsymbol{e\,b\,c} + \boldsymbol{c\,e\,b} + \boldsymbol{b\,e\,c})\,\boldsymbol{a\,e}^2\right\} \equiv 0, \tag{6.25}$$

which with (6.24) yields

$$\operatorname{tr} \boldsymbol{d\,c\,e\,b\,a\,e}^2 \equiv -\operatorname{tr} \boldsymbol{d\,b\,e\,c\,a\,e}^2. \tag{6.26}$$

Employing relations obtained from (6.26) by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, we see from the result (6.14), that, of the invariants formed by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in $\operatorname{tr} \boldsymbol{abecde}^2$, we may omit from the integrity basis all except

$$\operatorname{tr} \boldsymbol{a\,b\,e\,c\,d\,e}^2, \quad \operatorname{tr} \boldsymbol{b\,c\,e\,a\,d\,e}^2 \quad \text{and} \quad \operatorname{tr} \boldsymbol{c\,a\,e\,b\,d\,e}^2. \tag{6.27}$$

Taking all possible permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in the relations (6.18), we obtain, with the result (6.27), the relations

$$\begin{aligned}
&\operatorname{tr} \boldsymbol{a\,b\,c\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{d\,b\,c\,e\,a\,e}^2 \equiv \varphi_1,\\
&\operatorname{tr} \boldsymbol{a\,c\,b\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{d\,c\,b\,e\,a\,e}^2 \equiv \varphi_2,\\
&\operatorname{tr} \boldsymbol{c\,b\,a\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{d\,b\,a\,e\,c\,e}^2 \equiv \varphi_3,\\
&\operatorname{tr} \boldsymbol{a\,d\,b\,e\,c\,e}^2 + \operatorname{tr} \boldsymbol{c\,d\,b\,e\,a\,e}^2 \equiv \varphi_4,\\
&\operatorname{tr} \boldsymbol{b\,a\,c\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{d\,a\,c\,e\,b\,e}^2 \equiv \varphi_5,\\
&\operatorname{tr} \boldsymbol{b\,a\,d\,e\,c\,e}^2 + \operatorname{tr} \boldsymbol{c\,a\,d\,e\,b\,e}^2 \equiv \varphi_6,\\
&\operatorname{tr} \boldsymbol{a\,c\,d\,e\,b\,e}^2 + \operatorname{tr} \boldsymbol{b\,c\,d\,e\,a\,e}^2 \equiv \varphi_7,\\
&\operatorname{tr} \boldsymbol{a\,d\,c\,e\,b\,e}^2 + \operatorname{tr} \boldsymbol{b\,d\,c\,e\,a\,e}^2 \equiv \varphi_8,\\
&\operatorname{tr} \boldsymbol{c\,a\,b\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{d\,a\,b\,e\,c\,e}^2 \equiv \varphi_9,\\
&\operatorname{tr} \boldsymbol{a\,b\,d\,e\,c\,e}^2 + \operatorname{tr} \boldsymbol{c\,b\,d\,e\,a\,e}^2 \equiv \varphi_{10},\\
&\operatorname{tr} \boldsymbol{b\,c\,a\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{d\,c\,a\,e\,b\,e}^2 \equiv \varphi_{11},\\
&\operatorname{tr} \boldsymbol{b\,d\,a\,e\,c\,e}^2 + \operatorname{tr} \boldsymbol{c\,d\,a\,e\,b\,e}^2 \equiv \varphi_{12},
\end{aligned} \tag{6.28}$$

where the φ's are linear combinations of the invariants (6.27).

Not all of the relations (6.28) are independent. For example, adding the first and third of the relations (6.28), we have

$$\operatorname{tr} \boldsymbol{a\,b\,c\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{c\,b\,a\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{d\,b\,c\,e\,a\,e}^2 + \operatorname{tr} \boldsymbol{d\,b\,a\,e\,c\,e}^2 \equiv \varphi_1 + \varphi_3. \tag{6.29}$$

From Lemmas 1 and 2, we have

$$\operatorname{tr} \boldsymbol{c\,b\,a\,e\,d\,e}^2 = \operatorname{tr} \boldsymbol{e\,d\,e}^2\boldsymbol{c\,b\,a} = \operatorname{tr} \boldsymbol{a\,b\,c\,e}^2\boldsymbol{d\,e}. \tag{6.30}$$

Thus, with a relation of the form (3.2),

$$\operatorname{tr} \boldsymbol{a\,b\,c\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{c\,b\,a\,e\,d\,e}^2 = \operatorname{tr} \boldsymbol{a\,b\,c}(\boldsymbol{e\,d\,e}^2 + \boldsymbol{e}^2\boldsymbol{d\,e}) \equiv 0. \tag{6.31}$$

Introducing (6.31) into (6.29), we obtain

$$\operatorname{tr} \boldsymbol{d\,b\,c\,e\,a\,e}^2 + \operatorname{tr} \boldsymbol{d\,b\,a\,e\,c\,e}^2 \equiv \varphi_1 + \varphi_3. \tag{6.32}$$

Again, from relations of the form (3.2), together with Lemmas 1 and 2, we have

$$\begin{aligned} &\operatorname{tr} \boldsymbol{d\,b\,c\,e\,a\,e}^2 \equiv -\operatorname{tr} \boldsymbol{d\,b\,c\,e}^2\boldsymbol{a\,e} = -\operatorname{tr} \boldsymbol{e}^2\boldsymbol{a\,e\,d\,b\,c} = -\operatorname{tr} \boldsymbol{c\,b\,d\,e\,a\,e}^2, \\ &\text{and} \\ &\operatorname{tr} \boldsymbol{d\,b\,a\,e\,c\,e}^2 \equiv -\operatorname{tr} \boldsymbol{d\,b\,a\,e}^2\boldsymbol{c\,e} = -\operatorname{tr} \boldsymbol{e}^2\boldsymbol{c\,e\,d\,b\,a} = -\operatorname{tr} \boldsymbol{a\,b\,d\,e\,c\,e}^2. \end{aligned} \tag{6.33}$$

From (6.33) and (6.32), we have

$$\operatorname{tr} \boldsymbol{c\,b\,d\,e\,a\,e}^2 + \operatorname{tr} \boldsymbol{a\,b\,d\,e\,c\,e}^2 \equiv -\varphi_1 - \varphi_3, \tag{6.34}$$

which expresses the same fact as the tenth of the relations (6.28). Thus, the tenth of the relations (6.28) can be derived from the first and third. In a somewhat similar manner we can derive the twelfth relation from the fourth and eigth; we can derive the ninth relation from the fifth and sixth and the eleventh relation from the second and seventh. Thus, the last four of the relations (6.28) are derivable from the first eight.

By appropriate permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in (6.33), we obtain

$$\begin{aligned} \operatorname{tr} \boldsymbol{d\,c\,b\,e\,a\,e}^2 &\equiv -\operatorname{tr} \boldsymbol{b\,c\,d\,e\,a\,e}^2, \\ \operatorname{tr} \boldsymbol{a\,b\,c\,e\,d\,e}^2 &\equiv -\operatorname{tr} \boldsymbol{c\,b\,a\,e\,d\,e}^2, \\ \operatorname{tr} \boldsymbol{d\,a\,c\,e\,b\,e}^2 &\equiv -\operatorname{tr} \boldsymbol{c\,a\,d\,e\,b\,e}^2, \\ \text{and}\quad \operatorname{tr} \boldsymbol{b\,d\,c\,e\,a\,e}^2 &\equiv -\operatorname{tr} \boldsymbol{c\,d\,b\,e\,a\,e}^2. \end{aligned} \tag{6.35}$$

Substituting from (6.35) in the first eight of the relations (6.28), we obtain

$$\begin{aligned} -\operatorname{tr} \boldsymbol{c\,b\,a\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{d\,b\,c\,e\,a\,e}^2 &\equiv \varphi_1, \\ \operatorname{tr} \boldsymbol{a\,c\,b\,e\,d\,e}^2 - \operatorname{tr} \boldsymbol{b\,c\,d\,e\,a\,e}^2 &\equiv \varphi_2, \\ \operatorname{tr} \boldsymbol{c\,b\,a\,e\,d\,e}^2 + \operatorname{tr} \boldsymbol{d\,b\,a\,e\,c\,e}^2 &\equiv \varphi_3, \\ \operatorname{tr} \boldsymbol{a\,d\,b\,e\,c\,e}^2 + \operatorname{tr} \boldsymbol{c\,d\,b\,e\,a\,e}^2 &\equiv \varphi_4, \\ \operatorname{tr} \boldsymbol{b\,a\,c\,e\,d\,e}^2 - \operatorname{tr} \boldsymbol{c\,a\,d\,e\,b\,e}^2 &\equiv \varphi_5, \\ \operatorname{tr} \boldsymbol{b\,a\,d\,e\,c\,e}^2 + \operatorname{tr} \boldsymbol{c\,a\,d\,e\,b\,e}^2 &\equiv \varphi_6, \\ \operatorname{tr} \boldsymbol{a\,c\,d\,e\,b\,e}^2 + \operatorname{tr} \boldsymbol{b\,c\,d\,e\,a\,e}^2 &\equiv \varphi_7, \\ \operatorname{tr} \boldsymbol{a\,d\,c\,e\,b\,e}^2 - \operatorname{tr} \boldsymbol{c\,d\,b\,e\,a\,e}^2 &\equiv \varphi_8. \end{aligned} \tag{6.36}$$

Now, the twenty-four invariants obtained by permuting $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in $\operatorname{tr} \boldsymbol{abcede}^2$ are the twenty-four traces occurring on the left-hand sides of the relations (6.28). From (6.36) it is seen that each of the sixteen invariants occurring in the first eight of these relations is equivalent to some linear combination of

$$\operatorname{tr} \boldsymbol{b\,c\,d\,e\,a\,e}^2, \quad \operatorname{tr} \boldsymbol{c\,b\,a\,e\,d\,e}^2, \quad \operatorname{tr} \boldsymbol{c\,a\,d\,e\,b\,e}^2 \quad \text{and} \quad \operatorname{tr} \boldsymbol{c\,d\,b\,e\,a\,e}^2 \tag{6.37}$$

and the three invariants (6.27). By means of relations obtained from (6.35) by appropriate permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, we see that each of the traces appearing in the last four of the relations (6.28) is equivalent to the negative of one of those occurring in the first eight relations. We therefore have the result that each of the invariants obtained by permuting $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in $\operatorname{tr} \boldsymbol{abcede}^2$ is

equivalent to a linear combination of the invariants (6.37) and (6.27). Together with the result following from (6.11) and the result (6.27), we see that, of the invariants formed by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in $\operatorname{tr} \boldsymbol{abcede}^2$, $\operatorname{tr} \boldsymbol{abecde}^2$ and $\operatorname{tr} \boldsymbol{aebcde}^2$, we may omit from the integrity basis all except the invariants (6.37) and (6.27).

Consequently, of the invariants formed by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ in $\operatorname{tr} \boldsymbol{abcede}^2$, $\operatorname{tr} \boldsymbol{abecde}^2$ and $\operatorname{tr} \boldsymbol{aebcde}^2$, we may omit from the integrity basis all except the invariants (6.37), (6.27) and the invariants obtained by cyclic permutation of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ in these.

$\operatorname{tr} \boldsymbol{abcd}^2\boldsymbol{e}^2$. Finally, we consider the invariants formed from $\operatorname{tr} \boldsymbol{abcd}^2\boldsymbol{e}^2$ by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$.

Taking $\boldsymbol{y} = \boldsymbol{I}$ and $\boldsymbol{z} = \boldsymbol{d}^2\boldsymbol{e}^2$ in (3.1), we obtain

$$\operatorname{tr}\{(\boldsymbol{a\,b\,c} + \boldsymbol{a\,c\,b} + \boldsymbol{b\,c\,a} + \boldsymbol{b\,a\,c} + \boldsymbol{c\,b\,a} + \boldsymbol{c\,a\,b})\,\boldsymbol{d}^2\boldsymbol{e}^2\} \equiv 0. \tag{6.38}$$

Again, replacing $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ by $\boldsymbol{c}, \boldsymbol{d}^2, \boldsymbol{e}^2$ respectively in (3.1) and taking $\boldsymbol{y} = \boldsymbol{ab}$, $\boldsymbol{z} = \boldsymbol{I}$, we obtain

$$\operatorname{tr}\{\boldsymbol{a\,b}(\boldsymbol{c\,d}^2\boldsymbol{e}^2 + \boldsymbol{c\,e}^2\boldsymbol{d}^2 + \boldsymbol{d}^2\boldsymbol{c\,e}^2 + \boldsymbol{d}^2\boldsymbol{e}^2\boldsymbol{c} + \boldsymbol{e}^2\boldsymbol{d}^2\boldsymbol{c} + \boldsymbol{e}^2\boldsymbol{c\,d}^2)\} \equiv 0. \tag{6.39}$$

Further, replacing $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ by $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}^2$ respectively in (3.1) and taking $\boldsymbol{y} = \boldsymbol{ad}^2$ and $\boldsymbol{z} = \boldsymbol{I}$, we obtain

$$\operatorname{tr}\{\boldsymbol{a\,d}^2(\boldsymbol{b\,c\,e}^2 + \boldsymbol{b\,e}^2\boldsymbol{c} + \boldsymbol{c\,e}^2\boldsymbol{b} + \boldsymbol{c\,b\,e}^2 + \boldsymbol{e}^2\boldsymbol{b\,c} + \boldsymbol{e}^2\boldsymbol{c\,b})\} \equiv 0. \tag{6.40}$$

With appropriate changes of notation, we see from (3.3) that

$$\begin{aligned} &\operatorname{tr} \boldsymbol{a\,b\,d}^2\boldsymbol{c\,e}^2 \equiv 0, \quad \operatorname{tr} \boldsymbol{a\,b\,e}^2\boldsymbol{c\,d}^2 \equiv 0, \quad \operatorname{tr} \boldsymbol{a\,d}^2\boldsymbol{b\,c\,e}^2 \equiv 0, \\ &\operatorname{tr} \boldsymbol{a\,d}^2\boldsymbol{b\,e}^2\boldsymbol{c} \equiv 0, \quad \operatorname{tr} \boldsymbol{a\,d}^2\boldsymbol{c\,e}^2\boldsymbol{b} \equiv 0, \quad \operatorname{tr} \boldsymbol{a\,d}^2\boldsymbol{c\,b\,e}^2 \equiv 0. \end{aligned} \tag{6.41}$$

Using these relations in (6.39) and (6.40), we have, with Lemmas 1 and 2,

$$\operatorname{tr} \boldsymbol{a\,b\,c\,d}^2\boldsymbol{e}^2 + \operatorname{tr} \boldsymbol{c\,b\,a\,d}^2\boldsymbol{e}^2 + \operatorname{tr} \boldsymbol{c\,a\,b\,d}^2\boldsymbol{e}^2 + \operatorname{tr} \boldsymbol{b\,a\,c\,d}^2\boldsymbol{e}^2 \equiv 0 \tag{6.42}$$

and

$$\operatorname{tr} \boldsymbol{b\,c\,a\,d}^2\boldsymbol{e}^2 + \operatorname{tr} \boldsymbol{c\,b\,a\,d}^2\boldsymbol{e}^2 \equiv 0. \tag{6.43}$$

From (6.42) and (6.38), we have

$$\operatorname{tr} \boldsymbol{a\,c\,b\,d}^2\boldsymbol{e}^2 + \operatorname{tr} \boldsymbol{b\,c\,a\,d}^2\boldsymbol{e}^2 \equiv 0. \tag{6.44}$$

Interchanging $\boldsymbol{d}$ and $\boldsymbol{e}$ in (6.43) and employing Lemmas 1 and 2, we obtain

$$\operatorname{tr} \boldsymbol{a\,c\,b\,d}^2\boldsymbol{e}^2 + \operatorname{tr} \boldsymbol{a\,b\,c\,d}^2\boldsymbol{e}^2 \equiv 0. \tag{6.45}$$

Interchanging $\boldsymbol{a}$ and $\boldsymbol{b}$ in (6.43) and (6.45), we obtain

$$\begin{aligned} &\operatorname{tr} \boldsymbol{a\,c\,b\,d}^2\boldsymbol{e}^2 + \operatorname{tr} \boldsymbol{c\,a\,b\,d}^2\boldsymbol{e}^2 \equiv 0 \\ \text{and} \quad &\operatorname{tr} \boldsymbol{b\,c\,a\,d}^2\boldsymbol{e}^2 + \operatorname{tr} \boldsymbol{b\,a\,c\,d}^2\boldsymbol{e}^2 \equiv 0. \end{aligned} \tag{6.46}$$

From the relations (6.43) to (6.46), we see that each of the invariants formed from $\operatorname{tr} \boldsymbol{abcd}^2\boldsymbol{e}^2$ by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is equivalent either to $\operatorname{tr} \boldsymbol{abcd}^2\boldsymbol{e}^2$ or to $-\operatorname{tr} \boldsymbol{abcd}^2\boldsymbol{e}^2$. It follows that, of these invariants, we may omit from the integrity basis all except $\operatorname{tr} \boldsymbol{abcd}^2\boldsymbol{e}^2$. It is easily seen from Lemmas 1 and 2 that

$$\operatorname{tr} \boldsymbol{a\,b\,c\,e}^2\boldsymbol{d}^2 = \operatorname{tr} \boldsymbol{c\,b\,a\,d}^2\boldsymbol{e}^2,$$

with analogous results for each permutation of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Consequently, of the invariants formed from $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2\boldsymbol{e}^2$ and $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{e}^2\boldsymbol{d}^2$ by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, we may omit from the integrity basis all except $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2\boldsymbol{e}^2$.

Therefore, of the invariants formed from $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2\boldsymbol{e}^2$ by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$, we may omit from the integrity basis all except

$$\begin{array}{llll} \operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2\boldsymbol{e}^2, & \operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{d}\boldsymbol{c}^2\boldsymbol{e}^2, & \operatorname{tr} \boldsymbol{a}\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}^2\boldsymbol{e}^2, & \operatorname{tr} \boldsymbol{d}\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{e}^2, \\ \operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{e}\boldsymbol{c}^2\boldsymbol{d}^2, & \operatorname{tr} \boldsymbol{a}\boldsymbol{c}\boldsymbol{e}\boldsymbol{b}^2\boldsymbol{d}^2, & \operatorname{tr} \boldsymbol{c}\boldsymbol{b}\boldsymbol{e}\boldsymbol{a}^2\boldsymbol{d}^2, & \\ \operatorname{tr} \boldsymbol{a}\boldsymbol{d}\boldsymbol{e}\boldsymbol{b}^2\boldsymbol{c}^2, & \operatorname{tr} \boldsymbol{b}\boldsymbol{d}\boldsymbol{e}\boldsymbol{a}^2\boldsymbol{c}^2, & & \\ \operatorname{tr} \boldsymbol{c}\boldsymbol{d}\boldsymbol{e}\boldsymbol{a}^2\boldsymbol{b}^2. & & & \end{array} \tag{6.47}$$

Collecting the results obtained above, we see that an integrity basis for the five symmetric 3×3 matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ under the full or proper orthogonal group is formed from the integrity bases for the five sets of four matrices which can be selected from $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$, together with

(i) $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}\boldsymbol{e}$, $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{d}\boldsymbol{e}\boldsymbol{c}$, $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{e}\boldsymbol{c}\boldsymbol{d}$,
$\operatorname{tr} \boldsymbol{a}\boldsymbol{c}\boldsymbol{d}\boldsymbol{b}\boldsymbol{e}$, $\operatorname{tr} \boldsymbol{a}\boldsymbol{c}\boldsymbol{b}\boldsymbol{e}\boldsymbol{d}$, $\operatorname{tr} \boldsymbol{a}\boldsymbol{d}\boldsymbol{b}\boldsymbol{c}\boldsymbol{e}$;

(ii) $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{c}\boldsymbol{a}\boldsymbol{b}\boldsymbol{d}\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{c}\boldsymbol{d}\boldsymbol{a}\boldsymbol{b}\boldsymbol{e}^2$,
$\operatorname{tr} \boldsymbol{a}\boldsymbol{c}\boldsymbol{d}\boldsymbol{b}\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{d}\boldsymbol{a}\boldsymbol{c}\boldsymbol{b}\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{a}\boldsymbol{d}\boldsymbol{b}\boldsymbol{c}\boldsymbol{e}^2$

and invariants obtained from these by permuting $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ cyclically;

(iii) $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{e}\boldsymbol{c}\boldsymbol{d}\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{b}\boldsymbol{c}\boldsymbol{e}\boldsymbol{a}\boldsymbol{d}\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{c}\boldsymbol{a}\boldsymbol{e}\boldsymbol{b}\boldsymbol{d}\boldsymbol{e}^2$,
$\operatorname{tr} \boldsymbol{b}\boldsymbol{c}\boldsymbol{d}\boldsymbol{e}\boldsymbol{a}\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{c}\boldsymbol{b}\boldsymbol{a}\boldsymbol{e}\boldsymbol{d}\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{c}\boldsymbol{a}\boldsymbol{d}\boldsymbol{e}\boldsymbol{b}\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{c}\boldsymbol{d}\boldsymbol{b}\boldsymbol{e}\boldsymbol{a}\boldsymbol{e}^2$

and invariants obtained from these by permuting $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ cyclically;

(iv) $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{d}\boldsymbol{c}^2\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{a}\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}^2\boldsymbol{e}^2$, $\operatorname{tr} \boldsymbol{d}\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{e}^2$,
$\operatorname{tr} \boldsymbol{a}\boldsymbol{b}\boldsymbol{e}\boldsymbol{c}^2\boldsymbol{d}^2$, $\operatorname{tr} \boldsymbol{a}\boldsymbol{c}\boldsymbol{e}\boldsymbol{b}^2\boldsymbol{d}^2$, $\operatorname{tr} \boldsymbol{c}\boldsymbol{b}\boldsymbol{e}\boldsymbol{a}^2\boldsymbol{d}^2$,
$\operatorname{tr} \boldsymbol{a}\boldsymbol{d}\boldsymbol{e}\boldsymbol{b}^2\boldsymbol{c}^2$, $\operatorname{tr} \boldsymbol{b}\boldsymbol{d}\boldsymbol{e}\boldsymbol{a}^2\boldsymbol{c}^2$,
$\operatorname{tr} \boldsymbol{c}\boldsymbol{d}\boldsymbol{e}\boldsymbol{a}^2\boldsymbol{b}^2$.

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The Deformation of a Membrane Formed by Inextensible Cords

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Contents

1. Introduction

In earlier papers, Adkins & Rivlin (1955) and Adkins (1956a, 1956b) have considered the deformation of sheets and shells of a material which consists of incompressible isotropic elastic material reinforced with inextensible cords. Although the appropriate mathematical formalism can be developed for such problems, the presence of the elastic material makes it very difficult to solve any but the simplest problems. Furthermore, there are many situations of technological interest, in which the elastic material is either absent altogether or plays a secondary role in the determination of the deformational characteristics of the system. Accordingly, Rivlin (1955) considered the problem of plane strain in a net formed by two families of straight parallel inextensible cords.

In the present paper, we consider the deformation of a net, formed by two families of inextensible cords which, in the undeformed state, forms a surface not, in general, flat. The cords of the two families are assumed to form a curvilinear net on the surface and the angles of intersection of the cords of the two

families are assumed to be independent of position on the surface. The radius of curvature of the surface formed by the net, in both the deformed and undeformed states, is considered to be large compared with the intercepts formed by the adjacent cords of one family on a cord of the other family and this intercept is independent of the position on the surface and is the same for both families of cords. As in the earlier paper (RIVLIN 1955), it is considered that the cords are perfectly flexible, that the intersecting cords of the two families cannot move relative to each other at their points of intersection and that no two adjacent cords of a family are brought into contact as a result of the deformation.

The deformation of such a net by forces applied to it is treated as a membrane problem, the membrane having two directions of inextensibility at each point. Appropriate governing equations for the determination of the deformation and cord tensions are derived in §§ 2 to 6. In § 7, we simplify these relations in the case when the net forms a right-circular cylinder on which the bisectors of the angles between the two families of cords form generators and lines of latitude. The cylindrically symmetric deformation of such a net is then considered in § 8, in the case when the applied forces are normal to the net in its deformed state and, measured per unit area in the deformed state, are constant over the net, and the edges of the net are displaced axially. In § 9, we consider that the membrane is further surrounded by a smooth rigid coaxial cylinder, with which part of the net comes into contact when it is deformed.

In § 10, we consider a net formed by inextensible cords to be deformed in some known manner by specified forces and then subjected to a further infinitesimal deformation. The governing equations for this infinitesimal deformation are derived. In § 11, they are specialised to the case when the initial deformation has cylindrical symmetry.

In the problem considered in § 8, it is seen that the net remains undeformed under the action of the applied forces if the ends are held at a particular separation. In §§ 12 to 14, we consider that the net is held in this manner and its edges are then subjected to small displacements. The manner in which the resulting small displacements throughout the membrane may be calculated is discussed.

Finally, in § 15, we consider the case when small additional forces are applied to the membrane in the more general cylindrically symmetric state of deformation considered in § 8 and we discuss the manner in which the resulting small deformation may be calculated throughout the membrane.

2. The Equations of Equilibrium for a Membrane

We shall take as our starting point the equations of equilibrium for a membrane in the form given by GREEN & ZERNA (1954). It is assumed that the equation of the surface in its deformed state is known, in parametric form, in a curvilinear coordinate system x, the two parameters being* X^α, so that

$$x^i = x^i(X^\alpha) \tag{2.1}$$

* Throughout this paper we shall consider that Latin indices take the values 1, 2, 3 and Greek indices take the values 1, 2.

is the equation for the deformed surface. We shall denote by $n^{\alpha\beta}$ the contravariant stress components in the membrane in the surface coordinate system X defined by $X^1 = \text{const}$, $X^2 = \text{const}$. We shall assume also that the deformed membrane is acted on by tangential surface forces with contravariant components p^α per unit area in the coordinate system X and by normal surface tractions p per unit area, the areas being measured in the deformed state.

Let $\boldsymbol{r}$ be the position vector of a generic point P of the deformed membrane and let $\boldsymbol{n}$ denote the unit normal to the membrane at that point. Then, we shall define the quantities $b_{\alpha\beta}$ by the equation

$$d\boldsymbol{r}\cdot d\boldsymbol{n} = -\, b_{\alpha\beta}\, dX^\alpha\, dX^\beta. \tag{2.2}$$

With this notation the equations of equilibrium are given (see, for example, GREEN & ZERNA 1954) by

$$n^{\alpha\beta}_{,\alpha} + p^\beta = 0 \tag{2.3}$$

and

$$n^{\alpha\beta}\, b_{\alpha\beta} + p = 0, \tag{2.4}$$

where $_{,\alpha}$ denotes covariant derivation with respect to X^α, so that

$$n^{\alpha\beta}_{,\alpha} = \frac{\partial n^{\alpha\beta}}{\partial X^\alpha} + \Gamma^\alpha_{\alpha\lambda}\, n^{\lambda\beta} + \Gamma^\beta_{\alpha\lambda}\, n^{\alpha\lambda}, \tag{2.5}$$

where

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\lambda}\left(\frac{\partial g_{\lambda\beta}}{\partial X^\gamma} + \frac{\partial g_{\lambda\gamma}}{\partial X^\beta} - \frac{\partial g_{\beta\gamma}}{\partial X^\lambda}\right) \tag{2.6}$$

and $g_{\alpha\beta}$ is the covariant metric tensor for the coordinate system X. The contravariant metric tensor $g^{\alpha\beta}$ is defined by

$$g^{\alpha\gamma}\, g_{\gamma\beta} = \delta^\alpha_\beta, \tag{2.7}$$

where δ^α_β denotes the two-dimensional Kronecker delta. If y is an arbitrary three-dimensional rectangular Cartesian coordinate system, $g_{\alpha\beta}$ is given by

$$g_{\alpha\beta} = \frac{\partial y^k}{\partial X^\alpha}\,\frac{\partial y^k}{\partial X^\beta}. \tag{2.8}$$

Defining g by

$$g = g_{11}\, g_{22} - g_{12}^2, \tag{2.9}$$

we see from (2.7) that

$$g^{11} = g_{22}/g, \quad g^{22} = g_{11}/g \quad \text{and} \quad g^{12} = g^{21} = -\, g_{12}/g. \tag{2.10}$$

The components of the vector $d\boldsymbol{r}$ in the coordinate system y are given by

$$d\boldsymbol{r} = \left(\frac{\partial y^i}{\partial X^\alpha}\, dX^\alpha\right). \tag{2.11}$$

The components of the unit normal $\boldsymbol{n}$ in the system y may be obtained in the following manner. We note that the two vectors with components $(\partial y^i/\partial X^1)\, dX^1$ and $(\partial y^i/\partial X^2)\, dX^2$ respectively in the system y are both tangential to the membrane surface and are not parallel. Therefore, the vector with components

$$\varepsilon_{ijk}\,\frac{\partial y^j}{\partial X^1}\,\frac{\partial y^k}{\partial X^2}\, dX^1\, dX^2 \tag{2.12}$$

in the system y is parallel* to the vector $\boldsymbol{n}$. Since $\boldsymbol{n}$ has unit length it has components in the rectangular Cartesian coordinate system y given by

$$\boldsymbol{n} = \left[\varepsilon_{pqr}\,\varepsilon_{pmn}\frac{\partial y^q}{\partial X^1}\frac{\partial y^r}{\partial X^2}\frac{\partial y^m}{\partial X^1}\frac{\partial y^n}{\partial X^2}\right]^{-\frac{1}{2}}\left(\varepsilon_{ijk}\frac{\partial y^j}{\partial X^1}\frac{\partial y^k}{\partial X^2}\right). \tag{2.13}$$

Employing the relation

$$\varepsilon_{pqr}\,\varepsilon_{pmn} = \delta_{qm}\,\delta_{rn} - \delta_{pn}\,\delta_{rm}, \tag{2.14}$$

we see, with (2.8) and (2.9), that (2.13) may be re-written as

$$\boldsymbol{n} = \frac{1}{g^{\frac{1}{2}}}\left(\varepsilon_{ijk}\frac{\partial y^j}{\partial X^1}\frac{\partial y^k}{\partial X^2}\right). \tag{2.15}$$

We obtain, from (2.15),

$$d\boldsymbol{n} = \left(\frac{\partial}{\partial X^\beta}\left[\frac{1}{g^{\frac{1}{2}}}\varepsilon_{ijk}\frac{\partial y^j}{\partial X^1}\frac{\partial y^k}{\partial X^2}\right]d X^\beta\right). \tag{2.16}$$

Introducing the relations (2.16) and (2.11) into (2.2), we obtain

$$b_{\alpha\beta} = -\frac{\partial y^i}{\partial X^\alpha}\frac{\partial}{\partial X^\beta}\left[\frac{1}{g^{\frac{1}{2}}}\varepsilon_{ijk}\frac{\partial y^j}{\partial X^1}\frac{\partial y^k}{\partial X^2}\right]. \tag{2.17}$$

Equation (2.17) may be re-written as

$$b_{\alpha\beta} = -\varepsilon_{ijk}\left[\frac{\partial}{\partial X^\beta}\left\{\frac{1}{g^{\frac{1}{2}}}\frac{\partial y^i}{\partial X^\alpha}\frac{\partial y^j}{\partial X^1}\frac{\partial y^k}{\partial X^2}\right\} - \frac{1}{g^{\frac{1}{2}}}\frac{\partial y^j}{\partial X^1}\frac{\partial y^k}{\partial X^2}\frac{\partial^2 y^i}{\partial X^\alpha\,\partial X^\beta}\right]. \tag{2.18}$$

Since

$$\varepsilon_{ijk}\frac{\partial y^i}{\partial X^\alpha}\frac{\partial y^j}{\partial X^1}\frac{\partial y^k}{\partial X^2} = 0,$$

equation (2.18) yields

$$b_{\alpha\beta} = \varepsilon_{ijk}\frac{1}{g^{\frac{1}{2}}}\frac{\partial y^j}{\partial X^1}\frac{\partial y^k}{\partial X^2}\frac{\partial^2 y^i}{\partial X^\alpha\,\partial X^\beta}. \tag{2.19}$$

Denoting the physical components of stress in the coordinate system X by $t_{\alpha\beta}$, we have

$$\begin{aligned} n^{11} &= \left(\frac{g^{11}}{g_{11}}\right)^{\frac{1}{2}} t_{11}, \qquad & n^{22} &= \left(\frac{g^{22}}{g_{22}}\right)^{\frac{1}{2}} t_{22}, \\ n^{12} &= \left(\frac{g^{11}}{g_{22}}\right)^{\frac{1}{2}} t_{12} \quad \text{and} & n^{21} &= \left(\frac{g^{22}}{g_{11}}\right)^{\frac{1}{2}} t_{21}. \end{aligned} \tag{2.20}$$

Introducing the relations (2.10) into (2.20), we obtain

$$n^{11} = \left(\frac{g_{22}}{g\,g_{11}}\right)^{\frac{1}{2}} t_{11}, \qquad n^{22} = \left(\frac{g_{11}}{g\,g_{22}}\right)^{\frac{1}{2}} t_{22}$$

and

$$n^{12} = g^{21} = g^{-\frac{1}{2}} t_{12} = g^{-\frac{1}{2}} t_{21}. \tag{2.21}$$

The physical components f_α of the tangential surface traction per unit area are given by

$$f_1 = (g_{11})^{\frac{1}{2}} p^1 \quad \text{and} \quad f_2 = (g_{22})^{\frac{1}{2}} p^2. \tag{2.22}$$

Employing the relations (2.21) and (2.22) in equations (2.3) and (2.4) we can obtain the equations of equilibrium for the membrane in terms of physical components of stress and surface traction. In the next section this will be done in the particular case when the surface coordinate system X is an orthogonal one.

* The positive directions for X^1 and X^2 are chosen so that (2.12) is in the direction of the outward drawn normal to the surface.

3. The Equations of Equilibrium when the Surface Coordinate System is Orthogonal

If the coordinate system X is orthogonal, we have

$$g_{12} = 0. \tag{3.1}$$

Then, from (2.9) and (2.10), we obtain

$$g = g_{11} g_{22}, \quad g^{11} = 1/g_{11} \quad \text{and} \quad g^{22} = 1/g_{22}. \tag{3.2}$$

From (2.6), we then have, with (3.2),

$$\begin{aligned}
&\Gamma^1_{11} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial X^1} = \frac{1}{g^{\frac12}_{11}} \frac{\partial g^{\frac12}_{11}}{\partial X^1}, && \Gamma^2_{11} = -\frac{1}{2g_{22}} \frac{\partial g_{11}}{\partial X^2} = -\frac{g^{\frac12}_{11}}{g_{22}} \frac{\partial g^{\frac12}_{11}}{\partial X^2},\\
&\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial X^2} = \frac{1}{g^{\frac12}_{11}} \frac{\partial g^{\frac12}_{11}}{\partial X^2}, && \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial X^1} = \frac{1}{g^{\frac12}_{22}} \frac{\partial g^{\frac12}_{22}}{\partial X^1},\\
&\Gamma^1_{22} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial X^1} = -\frac{g^{\frac12}_{22}}{g_{11}} \frac{\partial g^{\frac12}_{22}}{\partial X^1} \quad \text{and} && \Gamma^2_{22} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial X^2} = \frac{1}{g^{\frac12}_{22}} \frac{\partial g^{\frac12}_{22}}{\partial X^2}.
\end{aligned} \tag{3.3}$$

From (2.3), (2.5) and (3.3), we obtain the two equations of equilibrium

$$\begin{aligned}
&\frac{\partial n^{11}}{\partial X^1} + \frac{\partial n^{12}}{\partial X^2} + \left(\frac{2}{g^{\frac12}_{11}} \frac{\partial g^{\frac12}_{11}}{\partial X^1} + \frac{1}{g^{\frac12}_{22}} \frac{\partial g^{\frac12}_{22}}{\partial X^1} \right) n^{11} + \\
&\qquad + \left(\frac{3}{g^{\frac12}_{11}} \frac{\partial g^{\frac12}_{11}}{\partial X^2} + \frac{1}{g^{\frac12}_{22}} \frac{\partial g^{\frac12}_{22}}{\partial X^2} \right) n^{12} - \frac{g^{\frac12}_{22}}{g_{11}} \frac{\partial g^{\frac12}_{22}}{\partial X^1} n^{22} + p^1 = 0\\
\text{and}\\
&\frac{\partial n^{21}}{\partial X^1} + \frac{\partial n^{22}}{\partial X^2} + \left(\frac{3}{g^{\frac12}_{22}} \frac{\partial g^{\frac12}_{22}}{\partial X^1} + \frac{1}{g^{\frac12}_{11}} \frac{\partial g^{\frac12}_{11}}{\partial X^1} \right) n^{12} + \\
&\qquad + \left(\frac{2}{g^{\frac12}_{22}} \frac{\partial g^{\frac12}_{22}}{\partial X^2} + \frac{1}{g^{\frac12}_{11}} \frac{\partial g^{\frac12}_{11}}{\partial X^2} \right) n^{22} - \frac{g^{\frac12}_{11}}{g_{22}} \frac{\partial g^{\frac12}_{11}}{\partial X^2} n^{11} + p^2 = 0.
\end{aligned} \tag{3.4}$$

From (2.21) and (3.2) we obtain

$$t_{11} = g_{11} n^{11}, \quad t_{22} = g_{22} n^{22} \quad \text{and} \quad t_{12} = t_{21} = (g_{11} g_{22})^{\frac12} n^{12}. \tag{3.5}$$

With (2.22) and (3.5), equations (3.4) may be re-written as

$$\begin{aligned}
&\frac{\partial (g^{\frac12}_{22} t_{11})}{\partial X^1} + \frac{\partial (g^{\frac12}_{11} t_{12})}{\partial X^2} + \frac{\partial g^{\frac12}_{11}}{\partial X^2} t_{12} - \frac{\partial g^{\frac12}_{22}}{\partial X^1} t_{22} + (g_{11} g_{22})^{\frac12} f_1 = 0\\
\text{and}\\
&\frac{\partial (g^{\frac12}_{22} t_{12})}{\partial X^1} + \frac{\partial (g^{\frac12}_{11} t_{22})}{\partial X^2} - \frac{\partial g^{\frac12}_{11}}{\partial X^2} t_{11} + \frac{\partial g^{\frac12}_{22}}{\partial X^1} t_{12} + (g_{11} g_{22})^{\frac12} f_2 = 0.
\end{aligned} \tag{3.6}$$

Employing equations (3.5) and (2.19) to substitute for $n^{\alpha\beta}$ and $b_{\alpha\beta}$ in (2.4), we obtain, with (3.2),

$$\begin{aligned}
\varepsilon_{ijk} \frac{\partial y^j}{\partial X^1} \frac{\partial y^k}{\partial X^2} \left[\frac{1}{g_{11}} \frac{\partial^2 y^i}{\partial X^1 \partial X^1} t_{11} + \frac{2}{(g_{11} g_{22})^{\frac12}} \frac{\partial^2 y^i}{\partial X^1 \partial X^2} t_{12} + \frac{1}{g_{22}} \frac{\partial^2 y^i}{\partial X^2 \partial X^2} t_{22} \right] + \\
+ p (g_{11} g_{22})^{\frac12} = 0.
\end{aligned} \tag{3.7}$$

Equation (3.7) may be re-written as

$$\begin{aligned}
&\varepsilon_{ijk}\frac{\partial y^j}{\partial x^p}\frac{\partial y^k}{\partial x^q}\frac{\partial x^p}{\partial X^1}\frac{\partial x^q}{\partial X^2}\times\\
&\times\left\{\frac{\partial y^i}{\partial x^m}\left[\frac{t_{11}}{g_{11}}\frac{\partial^2 x^m}{\partial X^1\partial X^1}+\frac{2t_{12}}{(g_{11}g_{22})^{\frac{1}{2}}}\frac{\partial^2 x^m}{\partial X^1\partial X^2}+\frac{t_{22}}{g_{22}}\frac{\partial^2 x^m}{\partial X^2\partial X^2}\right]+\right.\\
&\left.+\frac{\partial^2 y^i}{\partial x^m\partial x^n}\left[\frac{t_{11}}{g_{11}}\frac{\partial x^m}{\partial X^1}\frac{\partial x^n}{\partial X^1}+\frac{2t_{12}}{(g_{11}g_{22})^{\frac{1}{2}}}\frac{\partial x^m}{\partial X^1}\frac{\partial x^n}{\partial X^2}+\frac{t_{22}}{g_{22}}\frac{\partial x^m}{\partial X^2}\frac{\partial x^n}{\partial X^2}\right]\right\}+\\
&+p(g_{11}g_{22})^{\frac{1}{2}}=0
\end{aligned} \tag{3.8}$$

and in this equation of equilibrium, as well as in the equations of equilibrium (3.6), $g_{\alpha\beta}$ is given, from (2.8), by

$$g_{\alpha\beta}=G_{mn}\frac{\partial x^m}{\partial X^\alpha}\frac{\partial x^n}{\partial X^\beta}, \tag{3.9}$$

where G_{mn} is the covariant metric tensor of the coordinate system x.

4. The Deformation of a Membrane of Fabric—Kinematical Considerations

We now consider the deformation of a membrane of the ideal fabric described in § 1. We assume that in the undeformed state the distance between adjacent cords is h and that at each point of the membrane the cords of the two families are inclined at an angle 2α. We define the position of a point on the membrane in its undeformed state in the curvilinear orthogonal coordinate system X formed by the bisectors of the angles between the cords. The coordinates X^α of a point of the undeformed membrane in this system are then the distances of the point from an origin measured along the coordinate lines. We choose the X^1 axis so that the cords of the two families are inclined at angles α and $-\alpha$ to it.

Let us suppose that the membrane is deformed so that the point X^α moves to x^i in the curvilinear coordinate system x, the covariant metric tensor for which is G_{mn}.

If dS is the length of the element in the undeformed membrane joining the points X_α and $X_\alpha+dX_\alpha$ and ds is the corresponding element of length in the deformed membrane, we have

$$(dS)^2=dX^\alpha dX_\alpha=dX^\alpha dX^\alpha$$

and

$$(ds)^2=g_{\alpha\beta}\,dX^\alpha dX^\beta, \tag{4.1}$$

where $g_{\alpha\beta}$ is given by (3.9). From (4.1) we have

$$\left(\frac{ds}{dS}\right)^2=g_{\alpha\beta}L^\alpha L^\beta, \tag{4.2}$$

where L^α is the cosine of the angle between the linear element on the undeformed membrane and the direction of X^α at the point considered. If the linear element coincides with one or other of the cord directions at X^α, we have

$$L^1=\cos\alpha \quad\text{and}\quad L^2=\pm\sin\alpha. \tag{4.3}$$

Since the cords are inextensible, we have $(ds/dS)^2 = 1$ for L^α given by (4.3). Introducing this into (4.2), we obtain

$$g_{11}\cos^2\alpha + g_{22}\sin^2\alpha = 1$$

and

$$g_{12} = 0. \tag{4.4}$$

From (2.8), the second of equations (4.4) may be written as

$$\frac{\partial y^i}{\partial X^1}\frac{\partial y^i}{\partial X^2} = 0, \tag{4.5}$$

where y^i are the coordinates of a generic point of the membrane in an arbitrary rectangular Cartesian coordinate system y. Now, $\partial y^i/\partial X^1$ are proportional to the direction-cosines in the system y of the linear element in the deformed membrane which lies along the X^1 axis in the undeformed membrane and $\partial y^i/\partial X^2$ are proportional to the direction-cosines in the system y of the linear element in the deformed membrane which lies along the X^2 axis in the undeformed membrane. Thus, equation (4.5) and hence the second of equations (4.4) expresses the perpendicularity of these two elements.

We can also define the position of a point on the undeformed membrane with respect to the curvilinear net formed by the cords of the two families. If Z^α $(\alpha = 1, 2)$ denote distances measured along the cords of the net which are inclined at angles α and $-\alpha$ respectively to the direction of X^1, we have

$$dZ^1 = \frac{1}{2}\left(\frac{dX^1}{\cos\alpha} + \frac{dX^2}{\sin\alpha}\right) \quad\text{and}\quad dZ^2 = \frac{1}{2}\left(\frac{dX^1}{\cos\alpha} - \frac{dX^2}{\sin\alpha}\right).$$

Whence,

$$\frac{\partial}{\partial X^1} = \frac{1}{2\cos\alpha}\left(\frac{\partial}{\partial Z^1} + \frac{\partial}{\partial Z^2}\right) \quad\text{and}\quad \frac{\partial}{\partial X^2} = \frac{1}{2\sin\alpha}\left(\frac{\partial}{\partial Z^1} - \frac{\partial}{\partial Z^2}\right),$$

and

$$\frac{\partial}{\partial Z^1} = \cos\alpha\frac{\partial}{\partial X^1} + \sin\alpha\frac{\partial}{\partial X^2} \quad\text{and}\quad \frac{\partial}{\partial Z^2} = \cos\alpha\frac{\partial}{\partial X^1} - \sin\alpha\frac{\partial}{\partial X^2}. \tag{4.6}$$

We may also re-write the second of equations (4.4) as

$$\frac{\partial y^i}{\partial X^1}\left(\cos\alpha\frac{\partial y^i}{\partial X^1} + \sin\alpha\frac{\partial y^i}{\partial X^2}\right) = \frac{\partial y^i}{\partial X^1}\left(\cos\alpha\frac{\partial y^i}{\partial X^1} - \sin\alpha\frac{\partial y^i}{\partial X^2}\right). \tag{4.7}$$

With (4.6), (4.7) yields

$$\frac{\partial y^i}{\partial X^1}\frac{\partial y^i}{\partial Z^1} = \frac{\partial y^i}{\partial X^1}\frac{\partial y^i}{\partial Z^2}. \tag{4.8}$$

We now consider three elements of length at a point of the undeformed membrane. One of them lies along the X^1 direction and the remaining two have the same lengths and lie along the Z^1 and Z^2 directions. Equation (4.8) expresses the fact that in the deformed membrane the first of these bisects the angle between the remaining two.

We thus conclude that the curvilinear net on the deformed membrane defined by $X^1 = \text{const}$, $X^2 = \text{const}$ is an orthogonal net and at each point of the deformed membrane the coordinate lines bisect the angles between the cords. The directions of the coordinate lines are therefore principal directions for the deformation.

5. The Physical Components of Stress in the Deformed Membrane of Fabric

If τ_1 and τ_2 are the tensions in the cords which were initially inclined at angles α and $-\alpha$ respectively to the X^1 direction, we can easily show (see, for example, ADKINS & RIVLIN (1955) or RIVLIN (1955)) that the physical components of stress $t_{\alpha\beta}$ are given by

$$\begin{gathered} t_{11}=\frac{g_{11}^{\frac{1}{2}}\cos^2\alpha}{g_{22}^{\frac{1}{2}}h}(\tau_1+\tau_2), \qquad \tau_{22}=\frac{g_{22}^{\frac{1}{2}}\sin^2\alpha}{g_{11}^{\frac{1}{2}}h}(\tau_1+\tau_2) \\ t_{12}=t_{21}=\frac{\sin\alpha\cos\alpha}{h}(\tau_1-\tau_2). \end{gathered} \tag{5.1}$$

6. The Equations of Equilibrium for the Deformed Membrane of Fabric

The equations of equilibrium for the deformed membrane are given by (3.6) and (3.8). Introducing the expressions (5.1) for the physical components of stress into (3.6), we obtain

$$\begin{aligned} &\frac{\partial}{\partial X^1}\left[g_{11}^{\frac{1}{2}}(\tau_1+\tau_2)\right]\cos^2\alpha+\frac{\partial}{\partial X^2}\left[g_{11}^{\frac{1}{2}}(\tau_1-\tau_2)\right]\sin\alpha\cos\alpha+ \\ &\qquad+\frac{\partial g_{11}^{\frac{1}{2}}}{\partial X^2}(\tau_1-\tau_2)\sin\alpha\cos\alpha-\frac{g_{22}^{\frac{1}{2}}}{g_{11}^{\frac{1}{2}}}\frac{\partial g_{22}^{\frac{1}{2}}}{\partial X^1}(\tau_1+\tau_2)\sin^2\alpha+h(g_{11}g_{22})^{\frac{1}{2}}f_1=0 \end{aligned}$$

and (6.1)

$$\begin{aligned} &\frac{\partial}{\partial X^1}\left[g_{22}^{\frac{1}{2}}(\tau_1-\tau_2)\right]\sin\alpha\cos\alpha+\frac{\partial}{\partial X^2}\left[g_{22}^{\frac{1}{2}}(\tau_1+\tau_2)\right]\sin^2\alpha+ \\ &\qquad+\frac{\partial g_{22}^{\frac{1}{2}}}{\partial X^1}(\tau_1-\tau_2)\sin\alpha\cos\alpha-\frac{g_{11}^{\frac{1}{2}}}{g_{22}^{\frac{1}{2}}}\frac{\partial g_{11}^{\frac{1}{2}}}{\partial X^2}(\tau_1+\tau_2)\cos^2\alpha+h(g_{11}g_{22})^{\frac{1}{2}}f_2=0. \end{aligned}$$

These equations may be considerably simplified by making use of the first of the relations (4.4). Differentiating this relation with respect to X^1, we obtain

$$g_{11}^{\frac{1}{2}}\frac{\partial g_{11}^{\frac{1}{2}}}{\partial X^1}\cos^2\alpha=-g_{22}^{\frac{1}{2}}\frac{\partial g_{22}^{\frac{1}{2}}}{\partial X^1}\sin^2\alpha. \tag{6.2}$$

Introducing this result into the first of equations (6.1), we obtain

$$\frac{\partial}{\partial X^1}\left[g_{11}(\tau_1+\tau_2)\right]\cos^2\alpha+\frac{\partial}{\partial X^2}\left[g_{11}(\tau_1-\tau_2)\right]\sin\alpha\cos\alpha+h\,g_{11}g_{22}^{\frac{1}{2}}f_1=0. \tag{6.3}$$

Again, differentiating the first of equations (4.4) with respect to X^2 and employing the result so obtained in the second of equations (6.1), we obtain

$$\frac{\partial}{\partial X^1}\left[g_{22}(\tau_1-\tau_2)\right]\sin\alpha\cos\alpha+\frac{\partial}{\partial X^2}\left[g_{22}(\tau_1+\tau_2)\right]\sin^2\alpha+h\,g_{22}g_{11}^{\frac{1}{2}}f_2=0. \tag{6.4}$$

Employing the relations (4.6) in equations (6.3) and (6.4) we obtain

$$\frac{\partial}{\partial Z^1}(g_{11}\tau_1)+\frac{\partial}{\partial Z^2}(g_{11}\tau_2)+\frac{h}{\cos\alpha}g_{11}g_{22}^{\frac{1}{2}}f_1=0$$

and (6.5)

$$\frac{\partial}{\partial Z^1}(g_{22}\tau_1)-\frac{\partial}{\partial Z^2}(g_{22}\tau_2)+\frac{h}{\sin\alpha}g_{22}g_{11}^{\frac{1}{2}}f_2=0.$$

Introducing the expressions (5.1) for the physical components of stress into (3.8), we obtain

$$P\tau_1 + Q\tau_2 + R = 0, \tag{6.6}$$

where

$$\begin{aligned}
P = \varepsilon_{ijk} \frac{\partial y^j}{\partial x^p} \frac{\partial y^k}{\partial x^q} \frac{\partial x^p}{\partial X^1} \frac{\partial x^q}{\partial X^2} \times \\
\times \Bigg\{ \frac{\partial y^i}{\partial x^m} \left[\frac{\partial^2 x^m}{\partial X^1 \partial X^1} \cos^2\alpha + 2 \frac{\partial^2 x^m}{\partial X^1 \partial X^2} \sin\alpha\cos\alpha + \frac{\partial^2 x^m}{\partial X^2 \partial X^2} \sin^2\alpha \right] + \\
+ \frac{\partial^2 y^i}{\partial x^m \partial x^n} \left[\frac{\partial x^m}{\partial X^1} \frac{\partial x^n}{\partial X^1} \cos^2\alpha + 2 \frac{\partial x^m}{\partial X^1} \frac{\partial x^n}{\partial X^2} \sin\alpha\cos\alpha + \frac{\partial x^m}{\partial X^2} \frac{\partial x^n}{\partial X^2} \sin^2\alpha \right] \Bigg\}, \\
Q = \varepsilon_{ijk} \frac{\partial y^j}{\partial x^p} \frac{\partial y^k}{\partial x^q} \frac{\partial x^p}{\partial X^1} \frac{\partial x^q}{\partial X^2} \times \\
\times \Bigg\{ \frac{\partial y^i}{\partial x^m} \left[\frac{\partial^2 x^m}{\partial X^1 \partial X^1} \cos^2\alpha - 2 \frac{\partial^2 x^m}{\partial X^1 \partial X^2} \sin\alpha\cos\alpha + \frac{\partial^2 x^m}{\partial X^2 \partial X^2} \sin^2\alpha \right] + \\
+ \frac{\partial^2 y^i}{\partial x^m \partial x^n} \left[\frac{\partial x^m}{\partial X^1} \frac{\partial x^n}{\partial X^1} \cos^2\alpha - 2 \frac{\partial x^m}{\partial X^1} \frac{\partial x^n}{\partial X^2} \sin\alpha\cos\alpha + \frac{\partial x^m}{\partial X^2} \frac{\partial x^n}{\partial X^2} \sin^2\alpha \right] \Bigg\}
\end{aligned} \tag{6.7}$$

and

$$R = h\,p\,g_{11}\,g_{22}.$$

We have

$$\varepsilon_{ijk} \frac{\partial y^i}{\partial x^m} \frac{\partial y^j}{\partial x^p} \frac{\partial y^k}{\partial x^q} = \varepsilon_{mpq} \left| \frac{\partial y^k}{\partial x^l} \right|. \tag{6.8}$$

Employing this relation and the relations (4.7), we may re-write (6.7) as

$$\begin{aligned}
P = \varepsilon_{mpq} \left| \frac{\partial y^k}{\partial x^l} \right| \frac{\partial x^p}{\partial X^1} \frac{\partial x^q}{\partial X^2} \frac{\partial^2 x^m}{\partial Z^1 \partial Z^1} + \\
+ \varepsilon_{ijk} \frac{\partial y^j}{\partial x^p} \frac{\partial y^k}{\partial x^q} \frac{\partial x^p}{\partial X^1} \frac{\partial x^q}{\partial X^2} \frac{\partial^2 y^i}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial Z^1} \frac{\partial x^n}{\partial Z^1}, \\
Q = \varepsilon_{mpq} \left| \frac{\partial y^k}{\partial x^l} \right| \frac{\partial x^p}{\partial X^1} \frac{\partial x^q}{\partial X^2} \frac{\partial^2 x^m}{\partial Z^2 \partial Z^2} + \\
+ \varepsilon_{ijk} \frac{\partial y^j}{\partial x^p} \frac{\partial y^k}{\partial x^q} \frac{\partial x^p}{\partial X^1} \frac{\partial x^q}{\partial X^2} \frac{\partial^2 y^i}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial Z^2} \frac{\partial x^n}{\partial Z^2}
\end{aligned} \tag{6.9}$$

and

$$R = h\,p\,g_{11}\,g_{22}.$$

7. The Governing Equations for a Cylindrically Symmetric Deformation

We shall now assume that in the undeformed state, the membrane forms a right-circular cylinder and that the lines $X^2 = \text{const}$ are circumferential lines on the cylinder and the lines $X^1 = \text{const}$ are generators. We assume also that $X^2 = 0$ is the equatorial circle on the membrane. We shall take the coordinate system x^i to be a cylindrical polar system r, ϑ, z with the z-axis coinciding with the axis of the membrane, the origin lying in the plane of the equatorial circle and ϑ measured from the axial plane through $X^1 = 0$.

The covariant metric tensor G_{mn} for the coordinate system is then given by

$$\|G_{mn}\| = \left\| \begin{matrix} 1, & 0, & 0 \\ 0, & r^2, & 0 \\ 0, & 0, & 1 \end{matrix} \right\|. \tag{7.1}$$

We now assume that in the deformation a particle at X^α on the undeformed membrane moves to r, ϑ, z on the deformed membrane, where

$$r = r(X^2), \quad \vartheta = X^1/a \quad \text{and} \quad z = z(X^2). \tag{7.2}$$

Then, from (3.9) and (4.1), employing the notation $X^2 = \Sigma$, we have

$$g_{11} = (r/a)^2 \quad \text{and} \quad g_{22} = (dr/d\Sigma)^2 + (dz/d\Sigma)^2 = (d\sigma/d\Sigma)^2, \tag{7.3}$$

where σ denotes the curvilinear distance of the point considered on the deformed membrane from the equatorial circle, measured along a generator.

From (4.4) and (7.3), we obtain

$$\frac{r^2}{a^2}\cos^2\alpha + \left(\frac{d\sigma}{d\Sigma}\right)^2 \sin^2\alpha = 1, \tag{7.4}$$

and we note that the second of equations (4.4) is automatically satisfied.

We shall also assume that the applied force system is such that $f_1 = 0$ and f_2 and p are functions of Σ only. Then, from symmetry considerations we have

$$\tau_1 = \tau_2 = \tau \quad \text{(say)}, \tag{7.5}$$

which, with (5.1) and (7.3), yields

$$t_{11} = \frac{2\tau r \cos^2\alpha}{a h\, d\sigma/d\Sigma}, \quad t_{22} = \frac{2\tau a (d\sigma/d\Sigma)\sin^2\alpha}{h r} \quad \text{and} \quad t_{12} = 0. \tag{7.6}$$

Introducing equations (7.3) and (7.6) into equations (3.6) and bearing in mind that $f_1 = 0$ and, from symmetry considerations, τ is a function of X^2 only, we see that the first equation of equilibrium is automatically satisfied and the second takes the form

$$\frac{d}{d\sigma}(r t_{22}) - \frac{dr}{d\sigma} t_{11} + r f_2 = 0. \tag{7.7}$$

In the particular case when $f_2 = 0$, *i.e.* the forces applied to the membrane are normal to it in its deformed state, we can re-write equation (7.7) as

$$\frac{d}{dr}(r t_{22}) - t_{11} = 0. \tag{7.8}$$

The remaining equation of equilibrium may be obtained from equation (3.8) by introducing the relations (7.2), (7.3) and the last of the relations (7.6). In order to simplify the analysis we choose the rectangular Cartesian coordinate system y in such a way that the y^1, y^2, y^3 axes have radial, azimuthal and longitudinal directions respectively at the point of the deformed membrane considered. Then,

$$\frac{\partial y^1}{\partial x^1} = 1, \quad \frac{\partial y^2}{\partial x^2} = r, \quad \frac{\partial y^3}{\partial x^3} = 1$$

and

$$\frac{\partial y^i}{\partial x^j} = 0 \qquad (i \neq j). \tag{7.9}$$

Also,

$$\frac{\partial^2 y^1}{\partial x^2 \partial x^2} = r, \qquad \frac{\partial^2 y^2}{\partial x^1 \partial x^2} = 1$$

and, otherwise,

$$\frac{\partial^2 y^i}{\partial x^m \partial x^n} = 0. \tag{7.10}$$

In this manner, equation (3.8) becomes

$$\frac{1}{r}\frac{dz}{d\sigma} t_{11} + \frac{t_{22}}{(d\sigma/d\Sigma)^2}\left(\frac{dr}{d\sigma}\frac{d^2 z}{d\Sigma^2} - \frac{dz}{d\sigma}\frac{d^2 r}{d\Sigma^2}\right) = p. \tag{7.11}$$

Since the deformed membrane has cylindrical symmetry, the lines of latitude and longitude on the membrane are principal directions of curvature. Denoting the curvatures in the latitudinal and longitudinal directions by $\varkappa_1$ and $\varkappa_2$ respectively, we have

$$\varkappa_1 = \frac{1}{r}\frac{dz}{d\sigma} \quad \text{and} \quad \varkappa_2 = \frac{d}{dr}\left(\frac{dz}{d\sigma}\right). \tag{7.12}$$

Introducing the relations (7.12) into (7.11), we obtain

$$\varkappa_1 t_{11} + \varkappa_2 t_{22} = p. \tag{7.13}$$

Equations (7.8) and (7.13) are, of course, the well-known equations for the equilibrium of a membrane having cylindrical symmetry and deformed in a cylindrically symmetric manner by a cylindrically symmetric system of forces directed normally to it.

By substitution for t_{11} and t_{22} from (7.6) in (7.8) and (7.11), and employing the relation (7.4), we can re-write equations (7.8) and (7.11) in the forms

$$\left(\frac{d\sigma}{d\Sigma}\right)^2 \tau = \text{const}$$

and

$$(P+Q)\,\tau + R = 0, \tag{71.4}$$

where

$$P = Q = \frac{r}{a}\left[\left(\frac{dz}{d\Sigma}\frac{d^2 r}{d\Sigma^2} - \frac{dr}{d\Sigma}\frac{d^2 z}{d\Sigma^2}\right)\sin^2\alpha - \frac{r}{a^2}\frac{dz}{d\Sigma}\cos^2\alpha\right]$$

and

$$R = h\,p\left(\frac{r}{a}\frac{d\sigma}{d\Sigma}\right)^2. \tag{7.15}$$

These equations can also be obtained directly from (6.5), (6.6) and (6.7) by employing the relations (7.2), (7.3), (7.5), together with $f_1 = f_2 = 0$.

8. A Simple Case of Cylindrically Symmetric Deformation

In this section, we consider that the membrane initially forms a cylindrical surface of circular cross-section and is maintained in a cylindrically symmetric deformed state by a uniform normal force p per unit area, measured in the deformed state, its edges, which initially lie in the planes $\Sigma = \pm l$, being held in the planes $z = \pm c$ and with radii r_2, as shown in axial cross-section in Fig. 1. We denote by r_0 the radius of the equatorial circle in the deformed state and by r_1 the radius of the circle formed by the points on the deformed membrane for which $dz/dr = 0$.

From equations (7.6), we have

$$\frac{t_{22}}{t_{11}} = \frac{a^2}{r^2}\left(\frac{d\sigma}{d\Sigma}\right)^2 \tan^2\alpha . \tag{8.1}$$

Employing (8.1) to substitute for t_{22} in (7.8) and (7.13), we obtain

$$\frac{d}{dr}\left[t_{11}\frac{a^2}{r}\left(\frac{d\sigma}{d\Sigma}\right)^2 \tan^2\alpha\right] = t_{11}$$

and

$$\left[\varkappa_1 + \varkappa_2\frac{a^2}{r^2}\left(\frac{d\sigma}{d\Sigma}\right)^2 \tan^2\alpha\right] t_{11} = p . \tag{8.2}$$

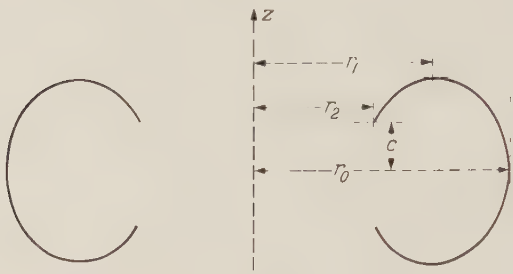

Fig. 1

Eliminating t_{11} from equations (8.2), we obtain with $dp/d\Sigma = 0$,

$$\frac{d}{dr}\left[\frac{r\left(\frac{d\sigma}{d\Sigma}\right)^2 \sin^2\alpha}{\varkappa_1\frac{r^2}{a^2}\cos^2\alpha + \varkappa_2\left(\frac{d\sigma}{d\Sigma}\right)^2\sin^2\alpha}\right] = \frac{\frac{r^2}{a^2}\cos^2\alpha}{\varkappa_1\frac{r^2}{a^2}\cos^2\alpha + \varkappa_2\left(\frac{d\sigma}{d\Sigma}\right)^2\sin^2\alpha} . \tag{8.3}$$

With (7.4), and employing the notation

$$\varphi = \frac{r\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)}{\varkappa_2 + (\varkappa_1 - \varkappa_2)\frac{r^2}{a^2}\cos^2\alpha} \tag{8.4}$$

equation (8.3) may be re-written as

$$\frac{d\varphi}{dr} = \frac{\frac{r}{a^2}\cos^2\alpha}{1 - \frac{r^2}{a^2}\cos^2\alpha}\,\varphi . \tag{8.5}$$

Equation (8.5) may be solved for φ to give

$$\varphi = \frac{A}{\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}} , \tag{8.6}$$

where A is an integration constant. From (8.4) and (8.6), we obtain, with the expressions (7.12) for $\varkappa_1$ and $\varkappa_2$,

$$\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\frac{d}{dr}\left(\frac{dz}{d\sigma}\right) + \frac{r}{a^2}\cos^2\alpha\,\frac{dz}{d\sigma} = \frac{1}{A}\,r\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{3}{2}} . \tag{8.7}$$

Solving this equation for $dz/d\sigma$, we obtain

$$\frac{dz}{d\sigma} = \left(\frac{1}{2A} r^2 + B\right)\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}, \tag{8.8}$$

where B is a constant of integration. Introducing into (8.8) the conditions that

$$\frac{dz}{dr} = \frac{dz}{d\sigma} = 0, \quad \text{when} \quad r = r_1$$

and

$$\frac{dz}{d\sigma} = 1 \qquad \text{when} \quad r = r_0, \tag{8.9}$$

we obtain

$$\left(\frac{1}{2A} r_0^2 + B\right)\left(1 - \frac{r_0^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}} = 1$$

and

$$\left(\frac{1}{2A} r_1^2 + B\right)\left(1 - \frac{r_1^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}} = 0. \tag{8.10}$$

Solving equations (8.10) for $2A$ and B, we obtain

$$B = -\frac{r_1^2}{2A} = -\frac{r_1^2}{(r_0^2 - r_1^2)\left(1 - \frac{r_0^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}}. \tag{8.11}$$

From (8.8) and (8.11), we obtain

$$\frac{dr}{d\sigma} = \mp\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}. \tag{8.12}$$

In this and the following equations, the upper sign is applicable to the region for which z is positive and the lower sign to that for which z is negative. From (8.12) we obtain

$$\sigma = \mp\int_{r_0}^{r} \frac{dr}{\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}, \tag{8.13}$$

the lower limit of the integration being taken as r_0 in order to satisfy the condition $\sigma = 0$, when $r = r_0$.

The equation, in the cylindrical polar coordinate system, describing the form of the deformed membrane is given, from (8.12), as

$$z = \mp\int_{r_0}^{r} \frac{\frac{1}{2A}(r^2 - r_1^2)\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}} dr}{\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}, \tag{8.14}$$

the lower limit in the integration again being taken as r_0 in order to satisfy the condition $z = 0$, when $r = r_0$.

If, as we initially assumed, $z = \pm c$ when $r = r_2$, we have, from (8.14),

$$c = -\int_{r_0}^{r_2} \frac{\frac{1}{2A}(r^2 - r_1^2)\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}} dr}{\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}. \tag{8.15}$$

From equations (7.4) and (8.12), we have

$$\frac{dr}{d\Sigma} = \mp \frac{1}{\sin\alpha}\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}, \quad (8.16)$$

whence

$$\Sigma = \mp \sin\alpha \int\limits_{r_0}^{r} \frac{dr}{\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}. \quad (8.17)$$

Since $\Sigma = \pm l$, when $r = r_2$, we have from (8.17)

$$l = \sin\alpha \int\limits_{r_0}^{r_2} \frac{dr}{\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}. \quad (8.18)$$

If r_0, r_1 and r_2 are specified, then equation (8.14) can be used to determine the shape of the deformed membrane and equations (8.18) and (8.15) may be used to determine the length of the undeformed membrane and the positions in which the edges of the deformed membrane should be held.

If, on the other hand, c, l and r_2 are specified, then equations (8.15) and (8.18) provide two equations for the determination of r_0 and r_1. Equation (8.14) may then be used, as before, to determine the shape of the deformed membrane.

If r_0 and r_1 are known, then from (8.8), with (8.11), we can determine $dz/d\sigma$ at each point of the deformed membrane and, from (7.12), the principal curvatures can be determined as

$$\varkappa_1 = \frac{1}{2Ar}(r^2 - r_1^2)\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}$$

and

$$\varkappa_2 = \frac{r}{2A}\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{-\frac{1}{2}}\left[2 - \frac{\cos^2\alpha}{a^2}(3r^2 - r_1^2)\right]. \quad (8.19)$$

From (8.2), (8.4), (8.6), (8.1) and (7.4), we obtain

$$t_{11} = \frac{A p \cos^2\alpha \frac{r}{a^2}}{\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{3}{2}}}$$

and

$$t_{22} = \frac{A p}{r\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}}. \quad (8.20)$$

From (7.6), we obtain, with (8.20)

$$\tau = \frac{h}{\sin 2\alpha}(t_{11} t_{22})^{\frac{1}{2}} = \frac{A p h}{2a\sin\alpha\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)}. \quad (8.21)$$

HOFFERBERTH (1956) has derived a formula analogous to (8.14) in the case when the cords are extensible. However, his expression for z takes the form of an integral in which the integrand contains the cord extension whose dependence on r is unknown. In the particular case when the cords are inextensible, the extension is zero and HOFFERBERTH's formula reduces to (8.14).

9. Further Development of the Problem

We now consider that the cylindrical membrane discussed in § 8 is located coaxially with a rigid circular cylindrical ring of radius r_0', so that when the membrane is deformed by the uniform normal force p, its edges being held at $r=r_2$, $z=\pm c$, it still assumes a form having cylindrical symmetry, but is partly in contact with the ring, as shown in axial cross-section in Fig. 2. We shall assume that there is no friction between the membrane and the restraining ring.

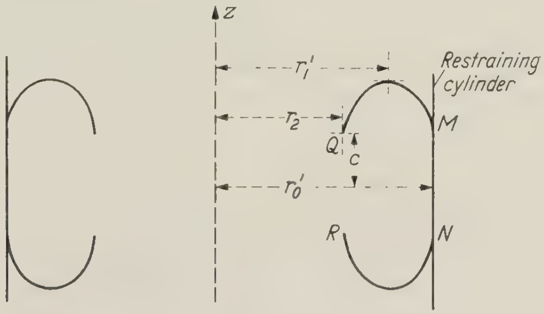

Fig. 2

For the region MN in contact with the restraining ring, we have

$$\varkappa_1 = 1/r_0' \quad \text{and} \quad \varkappa_2 = 0. \tag{9.1}$$

From (7.4) we then obtain, on MN,

$$\left(\frac{d\sigma}{d\Sigma}\right)^2 \sin^2\alpha = 1 - \frac{r_0'^2}{a^2} \cos^2\alpha. \tag{9.2}$$

From (9.2) and (7.6), we see that on MN

$$t_{11} = \frac{2\tau r_0' \cos^2\alpha \sin\alpha}{a h \left(1 - \frac{r_0'^2}{a^2} \cos^2\alpha\right)^{\frac{1}{2}}} \quad \text{and} \quad t_{22} = \frac{2\tau a}{h r_0'} \left(1 - \frac{r_0'^2}{a^2} \cos^2\alpha\right)^{\frac{1}{2}} \sin\alpha. \tag{9.3}$$

On the portions QM and RN of the membrane which are not in contact with the restraining cylinder, equations (8.1), (8.2) and (8.3) still apply, so we obtain, as in § 8, the result (8.8). It is easily seen, from NEWTON's third law, that at M and N the membrane is tangential to the restraining cylinder. We therefore have

$$\left.\begin{aligned} dz/dr &= 0 \quad \text{when} \quad r = r_1 \text{ (say)} \\ \text{and} \qquad dz/d\sigma &= 1 \quad \text{when} \quad r = r_0'. \end{aligned}\right\} \tag{9.4}$$

These relations replace (8.9) as the appropriate boundary conditions for the determination of A and B in (8.8), and we obtain

$$B = -\frac{r_1^2}{2A} = -\frac{r_1^2}{(r_0'^2 - r_1^2)\left(1 - \frac{r_0'^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}}. \tag{9.5}$$

If the z-coordinates for M and N are $\pm z_0'$, we obtain from (8.8), together with (9.5) and the fact that $dz/d\sigma = 1$ on MN,

$$\sigma \mp z_0' = \mp \int\limits_{r_0'}^{r} \frac{dr}{\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2 \left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}$$

and

$$z \mp z_0' = \mp \int\limits_{r_0'}^{r} \frac{\frac{1}{2A}(r^2 - r_1^2)\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}} dr}{\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2 \left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}, \tag{9.6}$$

the upper signs being applicable in the region QM and the lower signs in the region RN.

From (9.2) and $dz/d\sigma = 1$ on MN, we see that on MN

$$\frac{d\Sigma}{dz} = \sin\alpha \left(1 - \frac{r_0'^2}{a^2}\cos^2\alpha\right)^{-\frac{1}{2}}. \tag{9.7}$$

Thus,

$$\Sigma = \sin\alpha \left(1 - \frac{r_0'^2}{a^2}\cos^2\alpha\right)^{-\frac{1}{2}} z_0' \text{ at } M. \tag{9.8}$$

On QM and RN, equation (8.16) is still valid, so that, with (9.8),

$$\begin{aligned}\Sigma - \sin\alpha \left(1 - \frac{r_0'^2}{a^2}\cos^2\alpha\right)^{-\frac{1}{2}} z_0' \\ = \mp \sin\alpha \int\limits_{r_0'}^{r} \frac{dr}{\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}.\end{aligned} \tag{9.9}$$

We obtain immediately

$$\begin{aligned} l = \sin\alpha \left(1 - \frac{r_0'^2}{a^2}\cos^2\alpha\right)^{-\frac{1}{2}} z_0' - \\ - \sin\alpha \int\limits_{r_0'}^{r_2} \frac{dr}{\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}}\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}.\end{aligned} \tag{9.10}$$

Also, from the second of equations (9.6), we have

$$c - z_0' = \int\limits_{r_2}^{r_0'} \frac{\frac{1}{2A}(r^2 - r_1^2)\left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)^{\frac{1}{2}} dr}{\left[1 - \frac{1}{4A^2}(r^2 - r_1^2)^2 \left(1 - \frac{r^2}{a^2}\cos^2\alpha\right)\right]^{\frac{1}{2}}}. \tag{9.11}$$

Equations (9.10) and (9.11) provide two equations for the determination of r_1 and z_0' if c, l, r_0' and r_2 are specified. Equations (8.19), (8.20) and (8.21) are still valid for the determination of t_{11}, t_{22}, τ, $\varkappa_1$ and $\varkappa_2$ on QM and RN.

10. The Superposition of a Small Deformation on a Deformation of the Membrane

We now consider that the membrane discussed in § 6 is first subjected to a known deformation in which a generic particle initially at X^α in the coordinate system X moves to x^i in the curvilinear coordinate system x and surface tractions with physical components f_α and p, measured per unit area in this deformed state, act on the membrane. f_α are, of course, the components in the directions of the parametric curves $X^2 = \text{const}$ and $X^1 = \text{const}$ on the deformed surface and p is the component normal to the deformed membrane. The membrane may also be acted upon by edge tractions or subjected to edge displacements, but we shall not specify these in detail. This state of deformation will be called the *first deformed state.*

We consider further that the membrane is acted on by additional small forces, so that the generic particle undergoes an additional small displacement εu^i in the coordinate system x. The resulting state of deformation will be called the *second deformed state.* We shall employ the notation f^*_α and p^* for the physical components of the surface traction in the directions of the parametric curves on the membrane in the second deformed state and normal to it.

These are, of course, measured per unit area of the membrane in the second deformed state. We shall write

$$f^*_\alpha = f_\alpha + \varepsilon \bar{f}_\alpha \quad \text{and} \quad p^* = p + \varepsilon \bar{p}. \tag{10.1}$$

We shall assume throughout our analysis that ε is sufficiently small so that terms of first degree in ε may be neglected in comparison with terms of zero degree, terms of second degree in comparison with terms of first degree and so on.

In this section we shall derive the differential equations determining u^i. Writing

$$x^{*i} = x^i + \varepsilon u^i, \qquad G^*_{mn} = G_{mn} + \varepsilon \bar{G}_{mn}, \tag{10.2}$$

where G^*_{mn} is the covariant metric tensor for the coordinate system x at the point x^*_i, we apply equation (3.9) to the second deformed state of the membrane by replacing x^i and G_{mn} by x^{*i} and G^*_{mn} respectively. Denoting by $g^*_{\alpha\beta}$ the expression for $g_{\alpha\beta}$ then given by (3.9), and writing

$$g^*_{\alpha\beta} = g_{\alpha\beta} + \varepsilon \bar{g}_{\alpha\beta}, \tag{10.3}$$

we obtain, by equating coefficients of ε^0, the expression (3.9) for $g_{\alpha\beta}$ and, by equating coefficients of ε, the expression

$$\bar{g}_{\alpha\beta} = G_{mn}\left(\frac{\partial x^m}{\partial X^\alpha}\frac{\partial u^n}{\partial X^\beta} + \frac{\partial x^n}{\partial X^\beta}\frac{\partial u^m}{\partial X^\alpha}\right) + \frac{\partial x^m}{\partial X^\alpha}\frac{\partial x^n}{\partial X^\beta}\bar{G}_{mn}. \tag{10.4}$$

Applying equations (4.4) to the second deformed state of the body by replacing $g_{\alpha\beta}$ by $g^*_{\alpha\beta}$ and proceeding in a similar manner, we obtain equations (4.4) and

$$\bar{g}_{11}\cos^2\alpha + \bar{g}_{22}\sin^2\alpha = 0, \qquad \bar{g}_{12} = 0. \tag{10.5}$$

Equations (4.4) express the fact that the cords are not extended in the first deformation and equations (10.5) express the fact that they are not extended in the second (small) deformation.

We now consider the equations of equilibrium (6.5) and (6.6). Employing the notation τ_1, τ_2 and τ_1^*, τ_2^* for the tensions in the cords of the two families in the first and second deformed states respectively, we write

$$\tau_1^* = \tau_1 + \varepsilon\bar{\tau}_1 \quad \text{and} \quad \tau_2^* = \tau_2 + \varepsilon\bar{\tau}_2. \tag{10.6}$$

Then applying equations (6.5) to the second deformed state of the membrane by replacing $g_{11}, g_{22}, \tau_1, \tau_2, f_1$ and f_2 by $g_{11}^*, g_{22}^*, \tau_1^*, \tau_2^*, f_1^*$ and f_2^* respectively and making the substitutions (10.1), (10.3) and (10.6), we obtain equations (6.5) and the two equations

$$\frac{\partial}{\partial Z^1}(g_{11}\bar{\tau}_1) + \frac{\partial}{\partial Z^2}(g_{11}\bar{\tau}_2) + \frac{\partial}{\partial Z^1}(\tau_1\bar{g}_{11}) + \frac{\partial}{\partial Z^2}(\tau_2\bar{g}_{11}) + \\ + \frac{h}{\cos\alpha} g_{11} g_{22}^{\frac{1}{2}} f_1\left(\frac{\bar{g}_{11}}{g_{11}} + \frac{1}{2}\frac{\bar{g}_{22}}{g_{22}} + \frac{\bar{f}_1}{f_1}\right) = 0$$

and

$$\frac{\partial}{\partial Z^1}(g_{22}\bar{\tau}_1) - \frac{\partial}{\partial Z^2}(g_{22}\bar{\tau}_2) + \frac{\partial}{\partial Z^1}(\tau_1\bar{g}_{22}) - \frac{\partial}{\partial Z^2}(\tau_2\bar{g}_{22}) + \\ + \frac{h}{\sin\alpha} g_{22} g_{11}^{\frac{1}{2}} f_2\left(\frac{\bar{g}_{22}}{g_{22}} + \frac{1}{2}\frac{\bar{g}_{11}}{g_{11}} + \frac{\bar{f}_2}{f_2}\right) = 0. \tag{10.7}$$

Again, in (6.6) we replace P, Q, R, τ_1, τ_2 by P^*, Q^*, R^*, τ_1^* and τ_2^* for the second deformed state, where P^*, Q^* and R^* are defined by the expressions (6.7) in which x^i, y^i, p and $g_{\alpha\beta}$ are replaced by x^{*i}, y^{*i}, p^* and $g_{\alpha\beta}^*$, where y^{*i} denotes the coordinates in the rectangular Cartesian coordinate system y of a point at x^{*i} in the curvilinear coordinate system x. Then, with the notation (10.1), (10.2), (10.3), (10.6) and

$$P^* = P + \varepsilon\bar{P}, \qquad Q^* = Q + \varepsilon\bar{Q}, \qquad R^* = R + \varepsilon\bar{R}, \tag{10.8}$$

we obtain equations (6.6) and

$$P\bar{\tau}_1 + Q\bar{\tau}_2 + \tau_1\bar{P} + \tau_2\bar{Q} + \bar{R} = 0. \tag{10.9}$$

11. Small Deformation Superposed on a Cylindrically Symmetric Deformation

We shall now consider that the membrane has the form of of a right-circular cylinder of radius a in its undeformed state and that the bisectors of the angles between the cords lie in the circumferential and longitudinal directions on the membrane, as in § 7. We shall also assume that the first deformation is cylindrically symmetric and is described by equations (7.2) in the cylindrical polar coordinate system r, ϑ, z, and is maintained by a uniform surface traction p per unit area applied normally to the surface of the membrane in its first deformed state.

The fact that the cords are not extended in the first deformation is then expressed by equation (7.4) which, with the notation $dr/d\Sigma = r'$ and $dz/d\Sigma = z'$, may be re-written as

$$\frac{r^2}{a^2}\cos^2\alpha + (r'^2 + z'^2)\sin^2\alpha = 1. \tag{11.1}$$

The equations of equilibrium for the first deformed state are given by (7.14), which may be written as

$$(r'^2 + z'^2)\,\tau = \text{const} \tag{11.2}$$

and

$$2P\tau + R = 0, \tag{11.3}$$

where

$$P = \frac{r}{a}\left[(z' r'' - r' z'')\sin^2\alpha - \frac{r}{a^2} z' \cos^2\alpha\right]$$

and

$$R = h p \frac{r^2}{a^2}(r'^2 + z'^2) \tag{11.4}$$

and τ is the cord tension in the first deformed state.

Taking $x^i = r, \vartheta, z$ and employing the notation $u^i = u, v, w$, we see from (10.2) and (7.1) that

$$\overline{G}_{22} = 2 r u \quad \text{and} \quad \overline{G}_{mn} = 0 \quad (mn \neq 22). \tag{11.5}$$

From (10.4), (7.1), (7.2) and (11.5), we obtain, with the notation $X^1 = \Theta$ and $X^2 = \Sigma$,

$$\bar{g}_{11} = \frac{2r}{a}\left(r\frac{\partial v}{\partial \Theta} + \frac{u}{a}\right), \qquad \bar{g}_{22} = 2\left(r'\frac{\partial u}{\partial \Sigma} + z'\frac{\partial w}{\partial \Sigma}\right)$$

and

$$\bar{g}_{12} = r'\frac{\partial u}{\partial \Theta} + \frac{r^2}{a}\frac{\partial v}{\partial \Sigma} + z'\frac{\partial w}{\partial \Theta}. \tag{11.6}$$

Introducing (11.6) into (10.5), we obtain

$$\frac{r}{a}\left(\frac{u}{a} + r\frac{\partial v}{\partial \Theta}\right)\cos^2\alpha + \left(r'\frac{\partial u}{\partial \Sigma} + z'\frac{\partial w}{\partial \Sigma}\right)\sin^2\alpha = 0$$

and

$$r'\frac{\partial u}{\partial \Theta} + \frac{r^2}{a}\frac{\partial v}{\partial \Sigma} + z'\frac{\partial w}{\partial \Theta} = 0. \tag{11.7}$$

Since the first deformed state is maintained by a uniform pressure p per unit area applied normally to the membrane, we have

$$f_1 = f_2 = 0$$

and

$$\tau_1 = \tau_2 = \tau. \tag{11.8}$$

With (11.8), (7.3), (11.6) and (4.6) and, for convenience, using the symbols Z_1 and Z_2 in place of Z^1 and Z^2, equations (10.7) become

$$\frac{r^2}{a^2}\left(\frac{\partial \bar{\tau}_1}{\partial Z_1} + \frac{\partial \bar{\tau}_2}{\partial Z_2}\right) + \frac{2 r r'}{a^2}(\bar{\tau}_1 - \bar{\tau}_2)\sin\alpha +$$
$$+ 4\tau\frac{\partial}{\partial \Theta}\left(\frac{r u}{a^2} + \frac{r^2}{a}\frac{\partial v}{\partial \Theta}\right)\cos\alpha + \frac{h r^2}{a^2}(r'^2 + z'^2)^{\frac{1}{2}}\frac{\bar{f}_1}{\cos\alpha} = 0$$

and

$$(r'^2 + z'^2)\left(\frac{\partial \bar{\tau}_1}{\partial Z_1} - \frac{\partial \bar{\tau}_2}{\partial Z_2}\right) + 2(r' r'' + z' z'')(\bar{\tau}_1 + \bar{\tau}_2)\sin\alpha +$$
$$+ 4\tau\frac{\partial}{\partial \Sigma}\left(r'\frac{\partial u}{\partial \Sigma} + z'\frac{\partial w}{\partial \Sigma}\right)\sin\alpha + 4\left(r'\frac{\partial u}{\partial \Sigma} + z'\frac{\partial w}{\partial \Sigma}\right)\tau'\sin\alpha +$$
$$+ h(r'^2 + z'^2)\frac{r}{a}\frac{\bar{f}_2}{\sin\alpha} = 0, \tag{11.9}$$

where the notation $\tau' = d\tau/d\Sigma$ is used.

Again, we obtain from (10.9), with (11.8) and (11.4),

$$\tau(\overline{P}+\overline{Q})+\frac{r}{a}\left[(z'r''-r'z'')\sin^2\alpha-\frac{r}{a^2}z'\cos^2\alpha\right](\bar{\tau}_1+\bar{\tau}_2)+\overline{R}=0, \qquad (11.10)$$

where, from (6.7) and the definition of $\overline{P}$, $\overline{Q}$ and $\overline{R}$ given in § 10, we obtain

$$\begin{aligned}\overline{P}+\overline{Q}=&\frac{2rz'}{a}\left(\cos^2\alpha\frac{\partial^2 u}{\partial\Theta^2}+\sin^2\alpha\frac{\partial^2 u}{\partial\Sigma^2}\right)-\\&-\frac{2rr'}{a}\left(\cos^2\alpha\frac{\partial^2 w}{\partial\Theta^2}+\sin^2\alpha\frac{\partial^2 w}{\partial\Sigma^2}\right)+\frac{2u}{a}z'\left(r''\sin^2\alpha-\frac{2r}{a^2}\cos^2\alpha\right)+\\&+2\frac{\partial v}{\partial\Theta}z'r\left(r''\sin^2\alpha-\frac{3r}{a^2}\cos^2\alpha\right)+\\&+2\frac{\partial w}{\partial\Sigma}\frac{r}{a}\left(r''\sin^2\alpha-\frac{r}{a^2}\cos^2\alpha\right)-2\left(\frac{r}{a}\frac{\partial u}{\partial\Sigma}+\frac{r'u}{a}+rr'\frac{\partial v}{\partial\Theta}\right)z''\sin^2\alpha\end{aligned}$$

and

$$\begin{aligned}\overline{R}=2hp&\left[\frac{r^2}{a^2}\left(r'\frac{\partial u}{\partial\Sigma}+z'\frac{\partial w}{\partial\Sigma}\right)+\frac{r}{a}(r'^2+z'^2)\left(\frac{u}{a}+r\frac{\partial v}{\partial\Theta}\right)\right]+\\&+h\bar{p}\frac{r^2}{a^2}(r'^2+z'^2).\end{aligned} \qquad (11.11)$$

12. Small Deformation of a Right-Circular Cylindrical Membrane

Let us suppose that in the analysis of § 11, the first deformed state coincides with the undeformed state, but a uniform normal surface traction p is applied to the membrane. It is easy to see that this is a possible state of deformation if the edge conditions are appropriately chosen. Equations (7.2) become

$$r=a, \quad \vartheta=\Theta/a \quad \text{and} \quad z=\Sigma, \qquad (12.1)$$

so that equation (7.4) is satisfied. The first of the equations of equilibrium (7.14) is satisfied if $\tau=$const and the second is satisfied if the value of this constant is so chosen that

$$\tau=\frac{ahp}{2\cos^2\alpha}. \qquad (12.2)$$

It is thus apparent that a possible state of deformation of the cylindrical membrane is given by (12.1) if the edge displacements are zero, even if $p \neq 0$, and that the tensions in the cords are then given by (12.2). From (7.6) it then follows that the edge tractions must be longitudinal and given by $ap\tan^2\alpha$.

With (12.1), we obtain from (11.7)

$$\left(\frac{u}{a}+a\frac{\partial v}{\partial\Theta}\right)\cos^2\alpha+\frac{\partial w}{\partial\Sigma}\sin^2\alpha=0$$

and

$$a\frac{\partial v}{\partial\Sigma}+\frac{\partial w}{\partial\Theta}=0. \qquad (12.3)$$

Again, from (11.9) we obtain, with (12.1) and (12.2),

$$\frac{\partial\bar{\tau}_1}{\partial Z_1}+\frac{\partial\bar{\tau}_2}{\partial Z_2}+4\tau\left(\frac{1}{a}\frac{\partial u}{\partial\Theta}+a\frac{\partial^2 v}{\partial\Theta^2}\right)\cos\alpha+\frac{h\bar{f}_1}{\cos\alpha}=0$$

and

$$\frac{\partial\bar{\tau}_1}{\partial Z_1}-\frac{\partial\bar{\tau}_2}{\partial Z_2}+4\tau\frac{\partial^2 w}{\partial\Sigma^2}\sin\alpha+\frac{h\bar{f}_2}{\sin\alpha}=0. \qquad (12.4)$$

Also, from (11.10) and (11.11), we obtain

$$\tau(\bar{P}+\bar{Q}) - \frac{1}{a}(\bar{\tau}_1+\bar{\tau}_2)\cos^2\alpha + \bar{R} = 0, \tag{12.5}$$

with

$$\bar{P}+\bar{Q} = 2\left(\cos^2\alpha\,\frac{\partial^2 u}{\partial\Theta^2} + \sin^2\alpha\,\frac{\partial^2 u}{\partial\Sigma^2}\right) - 2\left(2\frac{u}{a^2} + 3\frac{\partial v}{\partial\Theta} + \frac{1}{a}\,\frac{\partial w}{\partial\Sigma}\right)\cos^2\alpha$$

and

$$\bar{R} = 2hp\left(\frac{u}{a} + a\frac{\partial v}{\partial\Theta} + \frac{\partial w}{\partial\Sigma}\right) + h\bar{p}. \tag{12.6}$$

13. Small Deformation of a Cylindrical Membrane with Specified Edge Displacements—Governing Equations

We now consider that the second deformation results from specified displacements of the edges of the shell at $\Sigma = \pm l$, and that $\bar{f}_\alpha = \bar{p} = 0$. Equations (12.4) then become

$$\frac{\partial\bar{\tau}_1}{\partial Z_1} + \frac{\partial\bar{\tau}_2}{\partial Z_2} + 4\tau\left(\frac{1}{a}\,\frac{\partial u}{\partial\Theta} + a\frac{\partial^2 v}{\partial\Theta^2}\right)\cos\alpha = 0$$

and

$$\frac{\partial\bar{\tau}_1}{\partial Z_1} - \frac{\partial\bar{\tau}_2}{\partial Z_2} + 4\tau\frac{\partial^2 w}{\partial\Sigma^2}\sin\alpha = 0 \tag{13.1}$$

and, in (12.5), $\bar{P}+\bar{Q}$ is given by the first of equations (12.6) and $\bar{R}$ is given by

$$\bar{R} = 2hp\left(\frac{u}{a} + a\frac{\partial v}{\partial\Theta} + \frac{\partial w}{\partial\Sigma}\right). \tag{13.2}$$

We can eliminate $u, w, \bar{\tau}_1$ and $\bar{\tau}_2$ from the five equations (12.3), (13.1) and (12.5) in the following manner.

Differentiating the first of equations (12.3) with respect to Θ and employing the second of equations (12.3) to substitute for $\partial w/\partial\Theta$, we obtain

$$\frac{\partial u}{\partial\Theta}\cos^2\alpha = -a^2\left(\frac{\partial^2 v}{\partial\Theta^2}\cos^2\alpha - \frac{\partial^2 v}{\partial\Sigma^2}\sin^2\alpha\right). \tag{13.3}$$

Employing the relation (13.3) in the first of equations (13.1), we obtain

$$\frac{\partial\bar{\tau}_1}{\partial Z_1} + \frac{\partial\bar{\tau}_2}{\partial Z_2} + 4\tau a\frac{\partial^2 v}{\partial\Sigma^2}\,\frac{\sin^2\alpha}{\cos\alpha} = 0. \tag{13.4}$$

With the second of equations (13.1), we obtain

$$\frac{\partial\bar{\tau}_1}{\partial Z_1} = -2\tau a\frac{\partial^2 v}{\partial\Sigma^2}\,\frac{\sin^2\alpha}{\cos\alpha} - 2\tau\frac{\partial^2 w}{\partial\Sigma^2}\sin\alpha$$

and

$$\frac{\partial\bar{\tau}_2}{\partial Z_2} = -2\tau a\frac{\partial^2 v}{\partial\Sigma^2}\,\frac{\sin^2\alpha}{\cos\alpha} + 2\tau\frac{\partial^2 w}{\partial\Sigma^2}\sin\alpha. \tag{13.5}$$

Differentiating the first of equations (13.5) with respect to Z_2 and the second with respect to Z_1 and adding the equations so obtained, we derive with (4.6),

$$\frac{\partial^2}{\partial Z_1\,\partial Z_2}(\bar{\tau}_1+\bar{\tau}_2) = 4\tau\sin^2\alpha\frac{\partial}{\partial\Sigma}\left(-a\frac{\partial^2 v}{\partial\Theta\,\partial\Sigma} + \frac{\partial^2 w}{\partial\Sigma^2}\right). \tag{13.6}$$

Now, differentiating (12.5) with respect to Θ, we obtain, with (13.2),

$$\tau\left(\frac{\partial\bar{P}}{\partial\Theta}+\frac{\partial\bar{Q}}{\partial\Theta}\right)-\frac{1}{a}\cos^2\alpha\left(\frac{\partial\bar{\tau}_1}{\partial\Theta}+\frac{\partial\bar{\tau}_2}{\partial\Theta}\right)+2hp\left(\frac{1}{a}\frac{\partial u}{\partial\Theta}+a\frac{\partial^2 v}{\partial\Theta^2}+\frac{\partial^2 w}{\partial\Theta\,\partial\Sigma}\right)=0. \quad (13.7)$$

Substituting in (13.7) for $\partial u/\partial\Theta$ from (13.3) and for $\partial w/\partial\Theta$ from the second of equations (12.3), we obtain

$$\tau\left(\frac{\partial\bar{P}}{\partial\Theta}+\frac{\partial\bar{Q}}{\partial\Theta}\right)-\frac{1}{a}\cos^2\alpha\left(\frac{\partial\bar{\tau}_1}{\partial\Theta}+\frac{\partial\bar{\tau}_2}{\partial\Theta}\right)+\frac{2hpa}{\cos^2\alpha}(\sin^2\alpha-\cos^2\alpha)\frac{\partial^2 v}{\partial\Sigma^2}=0. \quad (13.8)$$

Now, from (12.6),

$$\begin{aligned}\frac{\partial}{\partial\Theta}(\bar{P}+\bar{Q}) &= 2\frac{\partial}{\partial\Theta}\left(\frac{\partial^2 u}{\partial\Theta^2}\cos^2\alpha+\frac{\partial^2 u}{\partial\Sigma^2}\sin^2\alpha\right)-\\ &\quad -2\left(\frac{2}{a^2}\frac{\partial u}{\partial\Theta}+3\frac{\partial^2 v}{\partial\Theta^2}+\frac{1}{a}\frac{\partial^2 w}{\partial\Theta\,\partial\Sigma}\right)\cos^2\alpha.\end{aligned} \quad (13.9)$$

Substituting in (13.9) for $\partial u/\partial\Theta$ from (13.3) and for $\partial w/\partial\Theta$ from the second of equations (12.3), we obtain

$$\begin{aligned}&\frac{\partial}{\partial\Theta}(\bar{P}+\bar{Q})\\ &\quad=-\frac{2a^2}{\cos^2\alpha}\left(\frac{\partial^2}{\partial\Theta^2}\cos^2\alpha-\frac{\partial^2}{\partial\Sigma^2}\sin^2\alpha\right)\left(\frac{\partial^2}{\partial\Theta^2}\cos^2\alpha+\frac{\partial^2}{\partial\Sigma^2}\sin^2\alpha\right)v-\\ &\quad -2\frac{\partial^2 v}{\partial\Theta^2}\cos^2\alpha+2\frac{\partial^2 v}{\partial\Sigma^2}(\cos^2\alpha-2\sin^2\alpha).\end{aligned} \quad (13.10)$$

Forming the second derivative $\partial^2/\partial Z_1\partial Z_2$ of (13.8), introducing the results (13.10), (13.6), (12.3) and (12.2) and bearing in mind, from (4.6), that

$$\frac{\partial^2}{\partial Z_1\,\partial Z_2}=\cos^2\alpha\frac{\partial^2}{\partial\Theta^2}-\sin^2\alpha\frac{\partial^2}{\partial\Sigma^2}, \quad (13.11)$$

we obtain

$$\begin{aligned}&\left[a^2\left(\cos^2\alpha\frac{\partial^2}{\partial\Theta^2}-\sin^2\alpha\frac{\partial^2}{\partial\Sigma^2}\right)^2\left(\cos^2\alpha\frac{\partial^2}{\partial\Theta^2}+\sin^2\alpha\frac{\partial^2}{\partial\Sigma^2}\right)+\right.\\ &\quad\left.+\cos^4\alpha\left(\cos^2\alpha\frac{\partial^2}{\partial\Theta^2}-3\sin^2\alpha\frac{\partial^2}{\partial\Sigma^2}\right)\left(\frac{\partial^2}{\partial\Theta^2}+\frac{\partial^2}{\partial\Sigma^2}\right)\right]v=0.\end{aligned} \quad (13.12)$$

14. Small Deformation of a Cylindrical Membrane with Specified Edge Displacements—Solution of the Problem

If we exclude the possibility of dislocations or tears in the membrane, u, v and w must be continuous single-valued functions of position on the membrane. They must therefore be continuous periodic functions of Θ with period $2\pi a$. It follows that we may express u, v and w in the forms

$$u=\mathscr{R}\sum_{n=0}^{\infty}u_n e^{in\Theta/a},\qquad v=\mathscr{R}\sum_{n=0}^{\infty}v_n e^{in\Theta/a}$$

and

$$w=\mathscr{R}\sum_{n=0}^{\infty}w_n e^{in\Theta/a}, \quad (14.1)$$

where u_n, v_n and w_n are complex functions of Σ only.

We therefore consider a solution of (13.12) of the form

$$v = v_n e^{in\Theta/a}. \tag{14.2}$$

Substituting from (14.2) in (13.12), we obtain

$$\begin{aligned}\left(\frac{a}{n}\tan\alpha\right)^6 n^2 v_n^{(\mathrm{vi})} + \left(\frac{a}{n}\tan\alpha\right)^4 (n^2 - 3\cot^2\alpha)\, v_n^{(\mathrm{iv})} + \\ + \left(\frac{a}{n}\tan\alpha\right)^2 (3 - \cot^2\alpha - n^2)\, v_n'' + (1 - n^2)\, v_n = 0,\end{aligned} \tag{14.3}$$

where $v_n^{(\mathrm{vi})}$, $v_n^{(\mathrm{iv})}$ and v_n'' denote the sixth, fourth and second derivatives respectively of v_n with respect to Σ.

Corresponding to any solution for v of (13.12) of the form (14.2), we can obtain equations for u and w from (12.3). Substituting from (14.2) in (12.3), we see that

$$u = u_n e^{in\Theta/a} \quad \text{and} \quad w = w_n e^{in\Theta/a}, \tag{14.4}$$

where u_n and w_n are functions of Σ only, satisfying

$$\left(\frac{u_n}{a} + i n v_n\right)\cos^2\alpha + w_n' \sin^2\alpha = 0$$

and

$$a v_n' + \frac{in}{a} w_n = 0. \tag{14.5}$$

These equations yield

$$u_n = -ina\left(v_n + \frac{a^2}{n^2}\tan^2\alpha\, v_n''\right) \quad \text{and} \quad w_n = i\frac{a^2}{n} v_n', \tag{14.6}$$

unless $n = 0$.

The expressions for $\bar{\tau}_1$ and $\bar{\tau}_2$ corresponding to a solution for u, v and w of the form given in (14.2) and (14.4) may be obtained by taking

$$\bar{\tau}_1 = \bar{\tau}_n^{(1)} e^{in\Theta/a} \quad \text{and} \quad \bar{\tau}_2 = \bar{\tau}_n^{(2)} e^{in\Theta/a}, \tag{14.7}$$

in (13.5) and (12.5), where $\bar{\tau}_n^{(1)}$ and $\bar{\tau}_n^{(2)}$ are functions of Σ only, and employing the relations (4.6). From (13.5) we obtain

$$\frac{in}{a}\cos\alpha\, \bar{\tau}_n^{(1)} + \sin\alpha\, \bar{\tau}_n^{(1)\prime} = -2\tau a v_n'' \frac{\sin^2\alpha}{\cos\alpha} - 2\tau w_n'' \sin\alpha$$

and

$$\frac{in}{a}\cos\alpha\, \bar{\tau}_n^{(2)} - \sin\alpha\, \bar{\tau}_n^{(2)\prime} = -2\tau a v_n'' \frac{\sin^2\alpha}{\cos\alpha} + 2\tau w_n'' \sin\alpha. \tag{14.8}$$

From (12.5), we obtain, with (12.2), (12.6), (13.2), (14.2), (14.4), (14.7) and (14.6),

$$\begin{aligned}\frac{1}{a}\cos^2\alpha\, (\bar{\tau}_n^{(1)} + \bar{\tau}_n^{(2)}) &= hp\left(-\frac{n^2}{a}u_n + a\tan^2\alpha\, u_n'' - inv_n + w_n'\right) \\ &= inhp\left[(n^2-1)\, v_n + \frac{a^2}{n^2} v_n'' - \frac{a^4}{n^2}\tan^4\alpha\, v_n^{(\mathrm{iv})}\right].\end{aligned} \tag{14.9}$$

From (14.8) we have, with (14.6) and (12.2),

$$\frac{in}{a}\cos\alpha\, (\bar{\tau}_n^{(1)} - \bar{\tau}_n^{(2)}) + \sin\alpha\, (\bar{\tau}_n^{(1)\prime} + \bar{\tau}_n^{(2)\prime}) = -4\tau w_n'' \sin\alpha = -\frac{2ia^3hp\sin\alpha}{n\cos^2\alpha} v_n'''. \tag{14.10}$$

Substituting for $\bar{\tau}_n^{(1)} + \bar{\tau}_n^{(2)}$ from (14.9) in (14.10), we obtain

$$\frac{1}{a}\cos^2\alpha\,(\bar{\tau}_n^{(1)} - \bar{\tau}_n^{(2)}) = -a h p \tan\alpha \left[(n^2-1)\,v_n' + 3\,\frac{a^2}{n^2}\,v_n''' - \frac{a^4}{n^2}\tan^4\alpha\, v_n^{(\mathrm{v})}\right]. \tag{14.11}$$

Equations (14.9) and (14.11) may now be solved for $\bar{\tau}_n^{(1)}$ and $\bar{\tau}_n^{(2)}$ yielding

$$\begin{aligned} \frac{2}{a}\cos^2\alpha\,\bar{\tau}_n^{(1)} &= i n h p \left[(n^2-1)\,v_n + \frac{a^2}{n^2}\,v_n'' - \frac{a^4}{n^2}\tan^4\alpha\, v_n^{(\mathrm{iv})}\right] - \\ &\quad - a h p \tan\alpha \left[(n^2-1)\,v_n' + 3\,\frac{a^2}{n^2}\,v_n''' - \frac{a^4}{n^2}\tan^4\alpha\, v_n^{(\mathrm{v})}\right] \\ \text{and}\qquad & \\ \frac{2}{a}\cos^2\alpha\,\bar{\tau}_n^{(2)} &= i n h p \left[(n^2-1)\,v_n + \frac{a^2}{n^2}\,v_n'' - \frac{a^4}{n^2}\tan^4\alpha\, v_n^{(\mathrm{iv})}\right] + \\ &\quad + a h p \tan\alpha \left[(n^2-1)\,v_n' + 3\,\frac{a^2}{n^2}\,v_n''' - \frac{a^4}{n^2}\tan^4\alpha\, v_n^{(\mathrm{v})}\right]. \end{aligned} \tag{14.12}$$

We shall discuss here the solutions of the problem for $n=0$ and $n=1$ and leave discussion of the more general case for a later paper.

Case $n=0$. Equation (14.3) becomes

$$a^2 v_0^{(\mathrm{vi})} - 3\cot^4\alpha\, v_0^{(\mathrm{iv})} = 0. \tag{14.13}$$

Also, the second of equations (14.5) takes the form

$$v_0' = 0, \quad \text{whence} \quad v_0 = \text{const}, \tag{14.14}$$

which automatically satisfies (14.13). The remaining equation (14.5) now takes the form

$$\frac{u_0}{a}\cos^2\alpha + w_0'\sin^2\alpha = 0. \tag{14.15}$$

Also equations (14.8) take the forms

$$\bar{\tau}_0^{(1)\prime} = -2\tau w_0'' \quad \text{and} \quad \bar{\tau}_0^{(2)\prime} = -2\tau w_0''. \tag{14.16}$$

From (14.16) we obtain

$$\bar{\tau}_0^{(1)} = -2\tau w_0' + A_1 \quad \text{and} \quad \bar{\tau}_0^{(2)} = -2\tau w_0' + A_2, \tag{14.17}$$

where A_1 and A_2 are constants of integration. Also, introducing the result (14.14) and the expressions (14.2) and (14.4), with $n=0$, into (13.2) and the first of equations (12.6), we obtain

$$\begin{aligned} \bar{P} + \bar{Q} &= 2\sin^2\alpha\, u_0'' - 2\left(2\,\frac{u_0}{a^2} + \frac{w_0'}{a}\right)\cos^2\alpha \\ \text{and}\qquad & \\ \bar{R} &= 2hp\left(\frac{u_0}{a} + w_0'\right). \end{aligned} \tag{14.18}$$

Introducing (14.18), (14.17) and (12.2) into (12.5), we obtain

$$\sin^2\alpha\, u_0'' + \frac{3}{a}\cos^2\alpha\, w_0' = \frac{A}{2a\tau}\cos^2\alpha, \tag{14.19}$$

where $A = A_1 + A_2$. From (14.15) and (14.19), we obtain

$$\sin^4\alpha\, u_0'' - \frac{3}{a^2}\cos^4\alpha\, u_0 = \frac{A}{2a\,\tau}\cos^2\alpha\sin^2\alpha. \tag{14.20}$$

The general solution of this equation is

$$u_0 = A_0\, e^{(\sqrt{3}\cot^2\alpha)\,\Sigma/a} + B_0\, e^{-(\sqrt{3}\cot^2\alpha)\,\Sigma/a} - \frac{A\,a}{6\tau}\tan^2\alpha. \tag{14.21}$$

Introducing (14.21) into (14.15), we have

$$w_0' = -\frac{1}{a}\cot^2\alpha\left[A_0\, e^{(\sqrt{3}\cot^2\alpha)\,\Sigma/a} + B_0\, e^{-(\sqrt{3}\cot^2\alpha)\,\Sigma/a}\right] + \frac{A}{6\tau}. \tag{14.22}$$

Integrating equation (14.22), we obtain

$$w_0 = -\frac{1}{\sqrt{3}}A_0\, e^{(\sqrt{3}\cot^2\alpha)\,\Sigma/a} + \frac{1}{\sqrt{3}}B_0\, e^{-(\sqrt{3}\cot^2\alpha)\,\Sigma/a} + \frac{A}{6\tau}\Sigma + B, \tag{14.23}$$

where B is a constant of integration.

As an example, we consider the specific problem in which, in the deformation, each point of the edges $\Sigma = \pm l$ is displaced radially by a constant amount εU and parallel to the cylindrical axis by constant amounts $\pm\varepsilon W$. Taking $u = u_0$ and $w = w_0$ with A_0, B_0, A and B all real and introducing the boundary conditions $u_0 = U$ on $\Sigma = \pm l$ and $w_0 = \pm W$ on $\Sigma = \pm l$, we obtain

$$\begin{aligned}
U &= A_0\, e^{(\sqrt{3}\cot^2\alpha)\,l/a} + B_0\, e^{-(\sqrt{3}\cot^2\alpha)\,l/a} - \frac{A\,a}{6\tau}\tan^2\alpha,\\
U &= A_0\, e^{-(\sqrt{3}\cot^2\alpha)\,l/a} + B_0\, e^{(\sqrt{3}\cot^2\alpha)\,l/a} - \frac{A\,a}{6\tau}\tan^2\alpha,\\
W &= -\frac{1}{\sqrt{3}}A_0\, e^{(\sqrt{3}\cot^2\alpha)\,l/a} + \frac{1}{\sqrt{3}}B_0\, e^{-(\sqrt{3}\cot^2\alpha)\,l/a} + \frac{A}{6\tau}l + B\\
\text{and}\quad -W &= -\frac{1}{\sqrt{3}}A_0\, e^{-(\sqrt{3}\cot^2\alpha)\,l/a} + \frac{1}{\sqrt{3}}B_0\, e^{(\sqrt{3}\cot^2\alpha)\,l/a} - \frac{A}{6\tau}l + B.
\end{aligned} \tag{14.24}$$

These equations yield

$$\begin{aligned}
A_0 = B_0 &= \frac{l\,U + a\,W\tan^2\alpha}{2l\cosh\left[(\sqrt{3}\cot^2\alpha)\,l/a\right] - (2/\sqrt{3})\,a\tan^2\alpha\sinh\left[(\sqrt{3}\cot^2\alpha)\,l/a\right]},\\
A &= 6\tau\,\frac{2W\cosh\left[(\sqrt{3}\cot^2\alpha)\,l/a\right] + (2/\sqrt{3})\,U\sinh\left[(\sqrt{3}\cot^2\alpha)\,l/a\right]}{2l\cosh\left[(\sqrt{3}\cot^2\alpha)\,l/a\right] - (2/\sqrt{3})\,a\tan^2\alpha\sinh\left[(\sqrt{3}\cot^2\alpha)\,l/a\right]},
\end{aligned} \tag{14.25}$$

and $B = 0$. With these values for A_0, B_0, A and B the displacement components throughout the membrane are given by (14.21) and (14.23) by taking $u = u_0$ and $w = w_0$, together with $v = 0$.

Case $n = 1$. When $n = 1$ equation (14.3) becomes

$$(a\tan\alpha)^6\, v_1^{(\mathrm{vi})} + (a\tan\alpha)^4\,(1 - 3\cot^2\alpha)\, v_1^{(\mathrm{iv})} + (a\tan\alpha)^2\,(2 - \cot^2\alpha)\, v_1'' = 0. \tag{14.26}$$

This equation has the general solution

$$v_1 = A_1\, e^{\mu_1\Sigma/a\tan\alpha} + B_1\, e^{-\mu_1\Sigma/a\tan\alpha} + A_2\, e^{\mu_2\Sigma/a\tan\alpha} + B_2\, e^{-\mu_2\Sigma/a\tan\alpha} + A + B\,\Sigma, \tag{14.27}$$

where A_1, B_1, A_2, B_2, A and B are integration constants and μ_1^2 and μ_2^2 are solutions for λ of the equation

$$\lambda^2 + (1 - 3\cot^2\alpha)\,\lambda + (2 - \cot^2\alpha) = 0. \tag{14.28}$$

Then, from (14.27) and the second of equations (14.6), with $n=1$, we have

$$\begin{aligned} w_1 = i\,a\cot\alpha\,[\mu_1(A_1 e^{\mu_1 \Sigma/a\tan\alpha} - B_1 e^{-\mu_1\Sigma/a\tan\alpha}) + \\ + \mu_2(A_2 e^{\mu_2\Sigma/a\tan\alpha} - B_2 e^{-\mu_2\Sigma/a\tan\alpha})] + i\,a^2 B. \end{aligned} \tag{14.29}$$

From (14.27) and the first of equations (14.6), with $n=1$, we have

$$\begin{aligned} u_1 = -\,i\,a\,[(\mu_1^2+1)\,(A_1 e^{\mu_1\Sigma/a\tan\alpha} + B_1 e^{-\mu_1\Sigma/a\tan\alpha}) + \\ + (\mu_2^2+1)\,(A_2 e^{\mu_2\Sigma/a\tan\alpha} + B_2 e^{-\mu_2\Sigma/a\tan\alpha}) + (A + B\,\Sigma)]. \end{aligned} \tag{14.30}$$

As an example, we consider a problem in which the edges $\Sigma = l$ and $\Sigma = -l$ undergo equal and opposite small rigid motions. Let the rigid motion of the edge $\Sigma = l$ consist of rotations through angles $\varepsilon\omega_1$ and $\varepsilon\omega_2$ about the radial line through $\Theta = 0$ lying in the plane of the edge and the radial line through $\Theta = \frac{1}{2}\pi a$ lying in this plane and of translations εh_1 and εh_2 respectively in these directions. Then, on $\Sigma = l$, we have, neglecting terms of higher degree than the first in ε,

$$u = h_1\cos\frac{\Theta}{a} + h_2\sin\frac{\Theta}{a},$$

$$v = -\frac{h_1}{a}\sin\frac{\Theta}{a} + \frac{h_2}{a}\cos\frac{\Theta}{a} \tag{14.31}$$

and

$$w = a\left(\omega_1\sin\frac{\Theta}{a} - \omega_2\cos\frac{\Theta}{a}\right).$$

Thus, on $\Sigma = l$,

$$u = \mathscr{R}[(h_1 - i\,h_2)\,e^{i\Theta/a}],$$

$$v = \mathscr{R}\left[\frac{1}{a}(i\,h_1 + h_2)\,e^{i\Theta/a}\right] \tag{14.32}$$

and

$$w = \mathscr{R}[-\,a\,(i\,\omega_1 + \omega_2)\,e^{i\Theta/a}].$$

Similarly, on $\Sigma = -l$,

$$u = -\,\mathscr{R}[(h_1 - i\,h_2)\,e^{i\Theta/a}],$$

$$v = -\,\mathscr{R}\left[\frac{1}{a}(i\,h_1 + h_2)\,e^{i\Theta/a}\right] \tag{14.33}$$

and

$$w = -\,\mathscr{R}[-\,a\,(i\,\omega_1 + \omega_2)\,e^{i\Theta/a}].$$

Introducing

$$u_1 = h_1 - i\,h_2, \qquad v_1 = \frac{1}{a}(i\,h_1 + h_2)$$

$$\text{and} \quad w_1 = -\,a\,(i\,\omega_1 + \omega_2) \quad \text{on} \quad \Sigma = l$$

and

$$u_1 = -\,(h_1 - i\,h_2), \qquad v_1 = -\,\frac{1}{a}(i\,h_1 + h_2) \tag{14.34}$$

$$\text{and} \quad w_1 = a\,(i\,\omega_1 + \omega_2) \quad \text{on} \quad \Sigma = -l,$$

we obtain from (14.27), (14.29) and (14.30), with the notation

$$\psi_1 = \mu_1 l/a \tan\alpha \quad \text{and} \quad \psi_2 = \mu_2 l/a \tan\alpha, \tag{14.35}$$

$$\begin{aligned}
&-i a[(\mu_1^2+1)(A_1 e^{\psi_1} + B_1 e^{-\psi_1}) + \\
&\qquad + (\mu_2^2+1)(A_2 e^{\psi_2} + B_2 e^{-\psi_2}) + A + B l] = h_1 - i h_2, \\
&-i a[(\mu_1^2+1)(A_1 e^{-\psi_1} + B_1 e^{\psi_1}) + \\
&\qquad + (\mu_2^2+1)(A_2 e^{-\psi_2} + B_2 e^{\psi_2}) + A - B l] = -h_1 + i h_2, \\
&A_1 e^{\psi_1} + B_1 e^{-\psi_1} + A_2 e^{\psi_2} + B_2 e^{-\psi_2} + A + B l = \frac{1}{a}(i h_1 + h_2), \\
&A_1 e^{-\psi_1} + B_1 e^{\psi_1} + A_2 e^{-\psi_2} + B_2 e^{\psi_2} + A - B l = -\frac{1}{a}(i h_1 + h_2), \\
&i a \cot\alpha[\mu_1(A_1 e^{\psi_1} - B_1 e^{-\psi_1}) + \mu_2(A_2 e^{\psi_2} - B_2 e^{-\psi_2})] + \\
&\qquad\qquad + i a^2 B = -a(i\omega_1 + \omega_2)
\end{aligned} \tag{14.36}$$

and

$$\begin{aligned}
&i a \cot\alpha[\mu_1(A_1 e^{-\psi_1} - B_1 e^{\psi_1}) + \mu_2(A_2 e^{-\psi_2} - B_2 e^{\psi_2})] + \\
&\qquad\qquad + i a^2 B = a(i\omega_1 + \omega_2).
\end{aligned}$$

From (14.36) we obtain

$$\begin{aligned}
A_1 + B_1 &= \frac{\mu_2}{\mu_1 D}\tan\alpha(-\omega_1 + i\omega_2)\cosh\psi_2, \\
A_2 + B_2 &= \frac{\mu_1}{\mu_2 D}\tan\alpha(\omega_1 - i\omega_2)\cosh\psi_1, \\
A_1 - B_1 &= \frac{i\mu_2^2}{E}(h_1 - i h_2)\sinh\psi_2, \\
A_2 - B_2 &= -\frac{i\mu_1^2}{E}(h_1 - i h_2)\sinh\psi_1, \\
A &= \frac{1}{\mu_1\mu_2 D}\tan\alpha(\omega_1 - i\omega_2)(\mu_2^2 - \mu_1^2)\cosh\psi_1\cosh\psi_2, \\
a B\tan\alpha &= -\frac{i\mu_1\mu_2}{E}(h_1 - i h_2)(\mu_2\cosh\psi_1\sinh\psi_2 - \mu_1\sinh\psi_1\cosh\psi_2),
\end{aligned} \tag{14.37}$$

where D and E are defined by

and

$$\begin{aligned}
D &= \mu_2\cosh\psi_2\sinh\psi_1 - \mu_1\sinh\psi_2\cosh\psi_1, \\
E &= a(\mu_2^2 - \mu_1^2)\sinh\psi_1\sinh\psi_2 - \\
&\quad - \mu_1\mu_2 l\cot\alpha(\mu_2\cosh\psi_1\sinh\psi_2 - \mu_1\sinh\psi_1\cosh\psi_2).
\end{aligned} \tag{14.38}$$

15. Small Deformation Superposed on Cylindrically Symmetric Large Deformation

The governing equations in this case are equations (11.7), (11.9) and (11.10). We shall consider that $\bar{f}_\alpha$ and $\bar{p}$, in these equations, may be expressed as Fourier series in Θ with period $2\pi a$, the coefficients in which are functions of Σ only.

We accordingly write

$$\bar{f}_\alpha = \mathscr{R} \sum_{n=0}^{\infty} \bar{f}_n^{(\alpha)} e^{in\Theta/a} \quad \text{and} \quad \bar{p} = \mathscr{R} \sum_{n=0}^{\infty} \bar{p}_n e^{in\Theta/a}, \tag{15.1}$$

where $\bar{f}_n^{(\alpha)}$ and $\bar{p}_n$ are complex functions of Σ only. As in the case of the small deformation of a cylindrical membrane discussed in § 14, if we exclude the possibility of dislocations or tears in the membrane, u, v and w must be continuous periodic functions of Θ with period $2\pi a$. We can therefore express u, v and w in the forms

$$u = \sum_{n=0}^{\infty} u_n e^{in\Theta/a}, \quad v = \sum_{n=0}^{\infty} v_n e^{in\Theta/a} \quad \text{and} \quad w = \sum_{n=0}^{\infty} w_n e^{in\Theta/a}, \tag{15.2}$$

where u_n, v_n and w_n are complex functions of Σ only. We assume also that $\bar{\tau}_1$ and $\bar{\tau}_2$ may be similarly expressed by

$$\bar{\tau}_1 = \sum_{n-0}^{\infty} \bar{\tau}_n^{(1)} e^{in\Theta/a} \quad \text{and} \quad \bar{\tau}_2 = \sum_{n=0}^{\infty} \bar{\tau}_n^{(2)} e^{in\Theta/a}. \tag{15.3}$$

We therefore consider a solution of equations (11.7), (11.9) and (11.10), in which $\bar{f}_\alpha$ and $\bar{p}$ are given by

$$\bar{f}_\alpha = \bar{f}_n^{(\alpha)} e^{in\Theta/a} \quad \text{and} \quad \bar{p} = \bar{p}_n e^{in\Theta/a}, \tag{15.4}$$

which has the form

$$\begin{gathered} u = u_n e^{in\Theta/a}, \quad v = v_n e^{in\Theta/a}, \quad w = w_n e^{in\Theta/a}, \\ \bar{\tau}_1 = \bar{\tau}_n^{(1)} e^{in\Theta/a} \quad \text{and} \quad \bar{\tau}_2 = \bar{\tau}_n^{(2)} e^{in\Theta/a}. \end{gathered} \tag{15.5}$$

Introducing (15.5) into (11.7), we obtain

$$\begin{gathered} \frac{r}{a^2}(u_n + i n r v_n) \cos^2\alpha + (r' u_n' + z' w_n') \sin^2\alpha = 0 \\ \text{and} \qquad i n (r' u_n + z' w_n) + r^2 v_n' = 0. \end{gathered} \tag{15.6}$$

Again, introducing (15.4) and (15.5) into equations (11.9), and employing the relations (4.6) and the notation

$$\bar{\tau}_n^{(1)} + \bar{\tau}_n^{(2)} = \tau_n, \quad \bar{\tau}_n^{(1)} - \bar{\tau}_n^{(2)} = \sigma_n, \quad \tau_n' = d\tau_n/d\Sigma, \quad \sigma_n' = d\sigma_n/d\Sigma, \tag{15.7}$$

we obtain

$$\begin{aligned} & r\left(\frac{i n}{a} \tau_n \cos\alpha + \sigma_n' \sin\alpha\right) + 2 r' \sigma_n \sin\alpha + \\ & \qquad + 4\tau \frac{i n}{a}(u_n + i n r v_n) \cos\alpha + \frac{h r (r'^2 + z'^2)^{\frac{1}{2}}}{\cos\alpha} \bar{f}_n^{(1)} = 0 \\ \text{and} \quad & \\ & (r'^2 + z'^2)\left(\frac{i n}{a} \sigma_n \cos\alpha + \tau_n' \sin\alpha\right) + 2(r' r'' + z' z'') \tau_n \sin\alpha + \\ & \qquad + 4\tau \sin\alpha \frac{d}{d\Sigma}(r' u_n' + z' w_n') + 4\tau' \sin\alpha (r' u_n' + z' w_n') + \\ & \qquad + \frac{h (r'^2 + z'^2) r}{a \sin\alpha} \bar{f}_n^{(2)} = 0. \end{aligned} \tag{15.8}$$

Also, introducing (15.4) and (15.5) into (11.11), we obtain

$$\begin{aligned}\overline{P}+\overline{Q} = {} & \frac{2rz'}{a}\left(-\frac{n^2}{a^2}\cos^2\alpha\, u_n + \sin^2\alpha\, u_n''\right) + \\ & + \frac{2rr'}{a}\left(\frac{n^2}{a^2}\cos^2\alpha\, w_n - \sin^2\alpha\, w_n''\right) + \\ & + \frac{2z'}{a}\left(r''\sin^2\alpha - \frac{2r}{a^2}\cos^2\alpha\right) u_n + \\ & + \frac{2in}{a} z' r \left(r''\sin^2\alpha - \frac{3r}{a^2}\cos^2\alpha\right) v_n + \\ & + \frac{2r}{a}\left(r''\sin^2\alpha - \frac{r}{a^2}\cos^2\alpha\right) w_n' - \\ & - 2z''\sin^2\alpha\left(\frac{r}{a}u_n' + \frac{r'}{a}u_n + \frac{in}{a}rr'v_n\right)\end{aligned}$$

and

$$\begin{aligned}\overline{R} = {} & 2hp\frac{r}{a^2}\left[r(r'u_n' + z'w_n') + (r'^2+z'^2)(u_n + inrv_n)\right] + \\ & + h\overline{p}_n\frac{r^2}{a^2}(r'^2+z'^2).\end{aligned} \tag{15.9}$$

With these expressions for $\overline{P}+\overline{Q}$ and $\overline{R}$, we have equation (11.10), which, the notation (15.7), may be written

$$\tau(\overline{P}+\overline{Q}) + \frac{r}{a}\left[(z'r'' - r'z'')\sin^2\alpha - \frac{r}{a^2}z'\cos^2\alpha\right]\tau_n + \overline{R} = 0. \tag{15.10}$$

If r, z, $\overline{f}_n^{(1)}$, $\overline{f}_n^{(2)}$ and $\overline{p}_n$ are known functions of Σ, then the five equations (15.6), (15.8) and (15.10) can be used to determine u_n, v_n, w_n, τ_n and σ_n as functions of Σ throughout the membrane if appropriate boundary conditions are imposed on the values of these quantities. In this paper, we shall consider, as an example, the case when $\overline{f}_n^{(1)}=0$ and $\overline{f}_n^{(2)}$ and p_n are real functions of Σ. Then, from the symmetry of the problem, or otherwise, it is easily seen that u_n, w_n and τ_n are real, while v_n and σ_n are imaginary.

If u_n, u_n', w_n, w_n', τ_n and $i\sigma_n$ are known for some value of Σ $(=\Sigma_1$ say) then they can be determined at the neighboring point $\Sigma=\Sigma_1+\Delta\Sigma$ by the following procedure:

(i) from the values of u_n and u_n' at $\Sigma=\Sigma_1$ we can determine the value of u_n at $\Sigma=\Sigma_1+\Delta\Sigma$ from the formula

$$[u_n]_{\Sigma=\Sigma_1+\Delta\Sigma} = [u_n]_{\Sigma=\Sigma_1} + [u_n']_{\Sigma=\Sigma'}\Delta\Sigma;$$

(ii) in a similar manner, from the values of w_n and w_n' at $\Sigma=\Sigma_1$, we can determine the value of w_n at $\Sigma=\Sigma_1+\Delta\Sigma$;

(iii) from the first of equations (15.6), we can determine iv_n at $\Sigma=\Sigma_1$ and from the second of equations (15.6), we can determine iv_n' at $\Sigma=\Sigma_1$;

(iv) we then determine iv_n at $\Sigma=\Sigma_1+\Delta\Sigma$ from the values of iv_n and iv_n' at $\Sigma=\Sigma_1$;

(v) from the first of equations (15.8) we determine $i\sigma_n'$ at $\Sigma=\Sigma_1$;

(vi) from the values of $i\sigma_n$ and $i\sigma_n'$ at $\Sigma=\Sigma_1$, we calculate the value of $i\sigma_n$ at $\Sigma=\Sigma_1+\Delta\Sigma$;

(vii) differentiating the first of equations (15.6) with respect to Σ, we obtain

$$\frac{r'}{a^2}(u_n + 2inrv_n)\cos^2\alpha + (r''u_n' + z''w_n')\sin^2\alpha +$$
$$+ \frac{r}{a^2}(u_n' + inrv_n')\cos^2\alpha + (r'u_n'' + z'w_n'')\sin^2\alpha = 0,$$

which together with (15.10) provides a pair of equations for the determination of u_n'' and w_n'' at $\Sigma = \Sigma_1$;

(viii) from the values of u_n' and u_n'' at $\Sigma = \Sigma_1$, we calculate the value of u_n' at $\Sigma = \Sigma_1 + \Delta\Sigma$;

(ix) from the value of w_n' and w_n'' at $\Sigma = \Sigma_1$ we calculate w_n' at $\Sigma = \Sigma_1 + \Delta\Sigma$;

(x) from the second of equations (15.8), we calculate τ_n' at $\Sigma = \Sigma_1$;

(xi) from the values of τ_n and τ_n' at $\Sigma = \Sigma_1$, we calculate the value of τ_n at $\Sigma = \Sigma_1 + \Delta\Sigma$.

It is seen that if $u_n, u_n', w_n, w_n', \tau_n$ and $i\sigma_n$ are given on the edge $\Sigma = l$ of the membrane, then u_n, iv_n, w_n, τ_n and $i\sigma_n$ can be calculated throughout the membrane. If we obtain six linearly independent solutions of this kind determined by six different sets of values of $u_n, u_n', w_n, w_n', \tau_n$ and $i\sigma_n$ on $\Sigma = l$, we can, by linear combination of these solutions, construct a solution satisfying various other types of edge condition, e.g. u_n, v_n and w_n might be specified on $\Sigma = l$ and $\Sigma = -l$.

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